MATHEMATICAL PHYSICS SOLUTIONS

GATE-2010

Q1. Consider an anti-symmetric tensor P_{ij} with indices i and j running from 1 to 5. The number of independent components of the tensor is

(a) 3 (b) 10 (c) 9 (d) 6

Ans: (b)

Solution: The number of independent components of the tensor

$$= \frac{1}{2}\left(N^2 - N\right) = \frac{1}{2}\left(25 - 5\right) = 10 \qquad (\because\ N = 5)$$

Q2. The value of the integral $\oint_C \frac{e^z \sin(z)}{z^2} dz$, where the contour C is the unit circle: $|z-2| = 1$, is

(a) $2\pi i$ (b) $4\pi i$ (c) πi (d) 0

Ans: (d)

Solution: $\because\ |z-2| = 1 \Rightarrow 1 < z < 3$ i.e. the pole $z = 0$ does not lie inside the contour.

$\therefore\ \oint_C \frac{e^z \sin z}{z^2} dz = 2\pi i \times 0 = 0$.

Q3. The eigenvalues of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are

(a) 5, 2, -2 (b) -5, -1, -1 (c) 5, 1, -1 (d) -5, 1, 1

Ans: (c)

Solution: The characteristic equation of the matrix A, $|A - \lambda I| = 0$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 9\right] = 0 \Rightarrow \lambda = 1,\ 2-\lambda = \pm 3$$

$$\Rightarrow \lambda = 5, 1, -1$$

Q4. If $f(x) = \begin{cases} 0 & \text{for } x < 3, \\ x-3 & \text{for } x \geq 3 \end{cases}$ then the Laplace transform of $f(x)$ is

(a) $s^{-2}e^{3s}$ (b) $s^{2}e^{3s}$ (c) s^{-2} (d) $s^{-2}e^{-3s}$

Ans: (d)

Solution: $L\{f(x)\}=\int_0^{\infty}e^{-sx}f(x)dx=\int_0^{3}e^{-sx}f(x)dx+\int_3^{\infty}e^{-sx}f(x)dx=\int_3^{\infty}(x-3)e^{-sx}dx$

$$L\{f(x)\}=(x-3)\frac{e^{-sx}}{-s}\Big|_3^{\infty}-\int_3^{\infty}1\cdot\left(\frac{e^{-sx}}{-s}\right)dx=0+\frac{1}{s}\int_3^{\infty}e^{-sx}dx=\frac{1}{s}\left[\frac{e^{-sx}}{-s}\right]_3^{\infty}=s^{-2}e^{-3s}$$

Q5. The solution of the differential equation for $y(t)$: $\frac{d^2y}{dt^2}-y=2\cosh(t)$, subject to the initial conditions $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=0$, is

(a) $\frac{1}{2}\cosh(t)+t\sinh(t)$ (b) $-\sinh(t)+t\cosh(t)$

(c) $t\cosh(t)$ (d) $t\sinh(t)$

Ans: (d)

Solution: For C.F $(D^2-1)y=0\Rightarrow m=\pm1\Rightarrow C.F.=C_1e^{t}+C_2e^{-t}$

$$P.I.=\frac{1}{D^2-1}(2\cosh t)=\frac{1}{D^2-1}2\left(\frac{e^{t}+e^{-t}}{2}\right)=\frac{1}{D^2-1}(e^{t})+\frac{1}{D^2-1}(e^{-t})=\frac{t}{2}e^{t}+\frac{t}{2}(-e^{-t})$$

$$\Rightarrow y=C_1e^{t}+C_2e^{-t}+\frac{t}{2}e^{t}-\frac{t}{2}e^{-t}$$

As, $y(0)=0\Rightarrow C_1+C_2=0.....(1)$

$$\frac{dy}{dt}=C_1e^{t}-C_2e^{-t}+\frac{t}{2}e^{t}+\frac{1}{2}e^{t}+\frac{t}{2}e^{-t}-\frac{1}{2}e^{-t}$$

Also, $\left.\frac{dy}{dt}\right|_{t=0}=0\Rightarrow C_1-C_2+0+\frac{1}{2}+0-\frac{1}{2}=0\Rightarrow C_1-C_2=0.....(2)$

From equation (1) and (2),

$$C_1=0,C_2=0.$$

Thus $y=\frac{t}{2}e^{t}-\frac{t}{2}e^{-t}\Rightarrow y=t\sinh t$

GATE-2011

Q6. Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P. Which of the following statements is NOT TRUE?

(a) Det A = Det B (b) Trace of A = Trace of B

(c) A and B have the same eigenvectors (d) A and B have the same eigenvalues

Ans: (c)

Solution: If A and B be square matrices of the same type and if P be invertible matrix, then matrices A and $B = P^{-1}AP$ have the same characteristic roots.

Then, $B - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$ where I is identity matrix.

$$|B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| = |A - \lambda I||P^{-1}||P| = |A - \lambda I||PP^{-1}| = |A - \lambda I|$$

Thus, the matrices A and B (= $P^{-1}AP$) have the same characteristic equation and hence same characteristic roots or eigen values. Since, the sum of the eigen values of a matrix and product of eigen values of a matrix is equal to the determinant of matrix, hence third alternative is incorrect.

Q7. If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

(a) $\vec{\nabla} \times \vec{F} = 0$ (b) $\vec{\nabla} \cdot \vec{F} = 0$ (c) $\vec{\nabla} V = 0$ (d) $\nabla^2 V = 0$

Ans: (a)

Solution: Since, $\vec{F}$ is derivative of potential $V(r)$ and $\vec{F} = -\vec{\nabla} V(r)$

$$\Rightarrow \vec{\nabla} \times \vec{F} = -\vec{\nabla} \times (\vec{\nabla} V) = 0 .$$

Q8. A 3×3 matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to be positive integers. The largest eigenvalues of the matrix is

(a) 18 (b) 12 (c) 9 (d) 6

Ans: (d)

Solution: We know that for any matrix

1. The product of eigenvalues is equals to the determinant of that matrix.

2. $\lambda_1 + \lambda_2 + \lambda_3 + ... =$ Trace of matrix

$\lambda_1 + \lambda_2 + \lambda_3 = 11$ and $\lambda_1\lambda_2\lambda_3 = 36$. Hence, the largest eigen value of the matrix is 6.

Q9. The unit vector normal to the surface $x^2 + y^2 - z = 1$ at the point $P(1, 1, 1)$ is

(a) $\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ (b) $\frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$ (c) $\frac{\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$ (d) $\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$

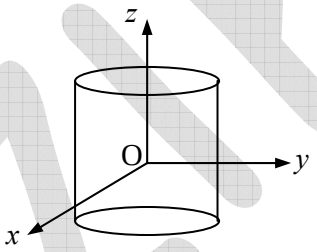
Ans: (d)

Solution: The equation of the system is $f(x,y,z) \equiv (x^2 + y^2 - z - 1) = 0$

$$\vec{\nabla} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 + y^2 - z - 1) = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Hence, unit normal vector at (1, 1, 1) $= \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$.

Q10. Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ be the position vector and $\hat{n}$ be a unit vector normal to the surface. The surface integral $\int_S \vec{r} \cdot \hat{n}\, ds$ over the closed surface of the cylinder is



(a) $2\pi a^2 (a + h)$ (b) $3\pi a^2 h$ (c) $2\,\pi a^2 h$ (d) zero

Ans: (b)

Solution: $\oint_S \vec{r}.\hat{n} ds = \int_V (\vec{\nabla}.\vec{r}) d\tau = 3\int_V d\tau = 3\pi a^2 h$

Q11. The solutions to the differential equation $\frac{dy}{dx} = -\frac{x}{y+1}$ are a family of

(a) circles with different radii

(b) circles with different centres

(c) straight lines with different slopes

(d) straight lines with different intercepts on the y-axis

Ans: (a)

Solution: $\frac{dy}{dx} = -\frac{x}{y+1} \Rightarrow xdx + ydy + dy = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + y = C_1 \Rightarrow x^2 + y^2 + 2y = 2C_1$

$\Rightarrow (x-0)^2+(y+1)^2 = 2C_1+1 = C$

which is a family of circles with different radii.

Q12. Which of the following statements is **TRUE** for the function $f(z)=\dfrac{z\sin z}{(z-\pi)^2}$?

(a) $f(z)$ is analytic everywhere in the complex plane

(b) $f(z)$ has a zero at $z=\pi$

(c) $f(z)$ has a pole of order 2 at $z=\pi$

(d) $f(z)$ has a simple pole at $z=\pi$

Ans: (c)

Solution: $f(z)=\dfrac{z\sin z}{(z-\pi)^2}$ has a pole of order 2 at $z=\pi$

Q13. Consider a counterclockwise circular contour $|z|=1$ about the origin. Let $f(z)=\dfrac{z\sin z}{(z-\pi)^2}$, then the integral $\oint f(z)dz$ over this contour is

(a) $-i\pi$ (b) zero (c) $i\pi$ (d) $2i\pi$

Ans: (b)

Solution: Since, pole $z=\pi$ does not lie inside the contour, hence

$$\oint f(z)dz = 0$$

GATE-2012

Q14. Identify the correct statement for the following vectors $\vec{a}=3\hat{i}+2\hat{j}$ and $\vec{b}=\hat{i}+2\hat{j}$

(a) The vectors $\vec{a}$ and $\vec{b}$ are linearly independent

(b) The vectors $\vec{a}$ and $\vec{b}$ are linearly dependent

(c) The vectors $\vec{a}$ and $\vec{b}$ are orthogonal

(d) The vectors $\vec{a}$ and $\vec{b}$ are normalized

Ans: (a)

Solution: If $\vec{a}=3\hat{i}+2\hat{j}$, $\vec{b}=\hat{i}+2\hat{j}$ are linearly dependent, then

$\vec{a} + m\vec{b} = 0$, for some values of m but here,

$3 + m = 0$ and $2 + 2m = 0$, do not have any solution. So, they are linearly independent.

$\vec{a} \cdot \vec{b} \neq 0$ (Not orthogonal); $\vec{a} \times \vec{b} \neq 0$ (Not normalized)

Q15. The number of independent components of the symmetric tensor A_{ij} with indices $i, j = 1, 2, 3$ is

(a) 1 (b) 3 (c) 6 (d) 9

Ans: (c)

Solution: For symmetric tensor, $A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$\because A_{12} = A_{21}, \quad A_{23} = A_{32}, \quad A_{13} = A_{31}$, hence there are six independent components.

Q16. The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are

(a) 0, 1, 1 (b) $0, -\sqrt{2}, \sqrt{2}$

(c) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$ (d) $\sqrt{2}, \sqrt{2}, 0$

Ans: (b)

Solution: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(\lambda^2 - 1) + \lambda = 0 \Rightarrow \lambda = 0, +\sqrt{2}, -\sqrt{2}$

GATE-2013

Q17. If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}\left(\vec{A} \cdot \left(\vec{B} \times \vec{r}\right)\right)$ is

(a) $\vec{A} \cdot \vec{B}$ (b) $\vec{A} \times \vec{B}$ (c) $\vec{r}$ (d) zero

Ans: (d)

Solution: Let $\vec{A} = A_0\left(\hat{x} + \hat{y} + \hat{z}\right)$, $\vec{B} = B_0\left(\hat{x} + \hat{y} + \hat{z}\right)$ and $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

$\vec{B} \times \vec{r} = \hat{x}(z - y)B_0 - \hat{y}(z - x)B_0 + \hat{z}(y - x)B_0 \Rightarrow \vec{\nabla}\left(\vec{A} \cdot \left(\vec{B} \times \vec{r}\right)\right) = 0$.

Q18. For the function $f(z)=\dfrac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z=1$ is (your answer should be an integer) _____.

Ans: 3

Solution: At $z=1$, pole is of order 2. So, residue is $\dfrac{1}{\underline{|2-1}}\dfrac{d^{2-1}}{dz^{2-1}}\left[\dfrac{(z-1)^2 16z}{(z+3)(z-1)^2}\right]_{z=1}=3$.

Q19. The degenerate eigenvalue of the matrix $\begin{bmatrix}4&-1&-1\\-1&4&-1\\-1&-1&4\end{bmatrix}$ is (your answer should be an integer) _____

Ans: $2,5,5$

$$\begin{bmatrix}4-\lambda&-1&-1\\-1&4-\lambda&-1\\-1&-1&4-\lambda\end{bmatrix}=0\Rightarrow(2-\lambda)\begin{bmatrix}1&-1&-1\\0&5-\lambda&0\\0&0&5-\lambda\end{bmatrix}=(2-\lambda)(5-\lambda)^2=0\Rightarrow\lambda=2,5,5.$$

Q20. The number of distinct ways of placing four indistinguishable balls into five distinguishable boxes is _____.

Ans: 120

Solution: $\underline{|4}\times C_4^5$=120 ways

GATE-2014

Q21. The unit vector perpendicular to the surface $x^2+y^2+z^2=3$ at the point (1, 1, 1) is

(a) $\dfrac{\hat{x}+\hat{y}-\hat{z}}{\sqrt{3}}$ (b) $\dfrac{\hat{x}-\hat{y}-\hat{z}}{\sqrt{3}}$ (c) $\dfrac{\hat{x}-\hat{y}+\hat{z}}{\sqrt{3}}$ (d) $\dfrac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}$

Ans: (d)

Solution: Let, $f=x^2+y^2+z^2-3=0\Rightarrow\vec{\nabla}f=2x\hat{x}+2y\hat{y}+2z\hat{z}$

$$\Rightarrow\hat{n}=\frac{\vec{\nabla}f}{|\vec{\nabla}f|}at(1,1,1)=\frac{2\hat{x}+2\hat{y}+2\hat{z}}{\sqrt{12}}=\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}$$

Q22. The matrix

$$A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$$ is

(a) orthogonal (b) symmetric (c) anti-symmetric (d) Unitary

Ans. : (d)

Solution: Unitary $A^{\dagger}A=I$

Q23. The value of the integral

$$\oint_C \frac{z^2}{e^z+1}dz$$

where C is the circle $|z|=4$, is

(a) $2\pi i$ (b) $2\pi^2 i$ (c) $4\pi^3 i$ (d) $4\pi^2 i$

Ans. : (c)

Solution: Pole $e^z=-1 \Rightarrow e^z=e^{i(2m+1)\pi}$ where $m=0,1,2,3.....$

For $z=i\pi$, $\text{Res}=\lim_{z=i\pi}\frac{\phi(z)}{\phi'(z)}=-\frac{\pi^2}{e^{i\pi}}=\pi^2$

Similarly, for $z=-i\pi, \text{Res}=\pi^2$

$\therefore I=2\pi i\left(\pi^2+\pi^2\right)=4\pi^3 i$

Q24. The solution of the differential equation $\frac{d^2y}{dt^2}-y=0$, subject to the boundary conditions

$y(0)=1$ and $y(\infty)=0$ is

(a) $\cos t+\sin t$ (b) $\cosh t+\sinh t$

(c) $\cos t-\sin t$ (d) $\cosh t-\sinh t$

Ans: (d)

Soluiton:

$D^2-1=0 \Rightarrow D=\pm 1 \Rightarrow y(t)=c_1e^t+c_2e^{-t}$

Applying boundary condition,

$y(0)=1 \Rightarrow 1=c_1+c_2$ and $y(\infty)=0 \Rightarrow 0=c_1e^{\infty}+c_2e^{-\infty} \Rightarrow c_1=0, c_2=1$

$\Rightarrow y(t)=e^{-t} \Rightarrow y(t)=\cosh t-\sinh t$

GATE-2015

Q25. Consider a complex function $f(z)=\dfrac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$. Which one of the following statements is correct?

(a) $f(z)$ has simple poles at $z=0$ and $z=-\dfrac{1}{2}$

(b) $f(z)$ has second order pole at $z=-\dfrac{1}{2}$

(c) $f(z)$ has infinite number of second order poles

(d) $f(z)$ has all simple poles

Ans.: (a)

Solution: $f(z)=\dfrac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$

For n^{th} order pole, Res. $=\lim_{z\to a}(z-a)^n f(z)=$ finite

At $z=0$, $\lim_{z\to 0} zf(z)=$ *finite* $\Rightarrow z=0$ is a simple pole.

At $z=-\dfrac{1}{2}$, $\lim_{z\to -\frac{1}{2}}\dfrac{\left(z+\frac{1}{2}\right)^2}{z\left(z+\frac{1}{2}\right)\cos z\pi}=\lim_{z\to -\frac{1}{2}}\dfrac{\left(z+\frac{1}{2}\right)}{z\cos z\pi}=\lim_{z\to -\frac{1}{2}}\dfrac{1}{1.\cos z\pi+z.\pi(-\sin z\pi)}$

$=\lim_{z\to -\frac{1}{2}}\dfrac{1}{\cos z\pi-z\pi\sin z\pi}=\dfrac{1}{-\frac{\pi}{2}}=-\dfrac{2}{\pi}=$ *finite*

$\Rightarrow f(z)$ has second order pole at $z=-\dfrac{1}{2}$

Q26. The value of $\int_0^3 t^2\delta(3t-6)dt$ is_____ (upto one decimal place)

Ans.: 1.33

Solution: $\int_0^3 t^2\delta(3t-6)\,dt=\int_0^3 t^2\delta\left[3(t-2)\right]dt=\dfrac{1}{3}\int_0^3 t^2\delta(t-2)\,dt=\dfrac{4}{3}$

Q27. If $f(x)=e^{-x^2}$ and $g(x)=|x|e^{-x^2}$, then

(a) f and g are differentiable everywhere

(b) f is differentiable everywhere but g is not

(c) g is differentiable everywhere but f is not

(d) g is discontinuous at $x=0$

Ans. (b)

Solution: $f(x)=e^{-x^2}$ is differentiable but $g(x)=|x|e^{-x^2}$ is not differentiable.

$$g(x)=\begin{cases}-xe^{-x^2}; & x<0\\ xe^{-x^2}; & x>0\end{cases}$$

Left hand Limit $\lim_{h\to 0} g(x-h)=-(x-h)e^{-(x-h)^2}$

Right hand Limit $\lim_{h\to 0} g(x+h)=(x+h)e^{-(x+h)^2}$

$\Rightarrow \lim_{h\to 0} g(x-h)\neq \lim_{h\to 0} g(x+h)$

Q28. Consider $w=f(z)=u(x,y)+iv(x,y)$ to be an analytic function in a domain D. Which one of the following options is NOT correct?

(a) $u(x,y)$ satisfies Laplace equation in D

(b) $v(x,y)$ satisfies Laplace equation in D

(c) $\int_{z_1}^{z_2} f(z)dz$ is dependent on the choice of the contour between z_1 and z_2 in D

(d) $f(z)$ can be Taylor expended in D

Ans.: (c)

Solution: $w=f(z)=u(x,y)+iv(x,y)$ to be an analytic function in a domain D, $\int_{z_1}^{z_2} f(z)dz$ is independent of the choice of the contour between z_1 and z_2 in D.

Q29. The Heaviside function is defined as $H(t)=\begin{cases}+1, & \text{for} \quad t>0\\ -1, & \text{for} \quad t<0\end{cases}$ and its Fourier transform is given by $-2i/\omega$. The Fourier transform of $\frac{1}{2}\left[H(t+1/2)-H(t-1/2)\right]$ is

(a) $\dfrac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$ (b) $\dfrac{\cos\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$ (c) $\sin\left(\frac{\omega}{2}\right)$ (d) 0

Ans.: (a)

Solution: $H(f)=\int_{-\infty}^{\infty}H(t)e^{-i2\pi ft}dt$, for a function $H(t)$, $H(f)=-\dfrac{2i}{\omega}$

For $H(t-t_0)$, Fourier Transform is $e^{-i2\pi ft_0}H(f)$

Shifting Theorem

For $\frac{1}{2}\left[H\left(t+\frac{1}{2}\right)-H\left(t-\frac{1}{2}\right)\right]=\frac{1}{2}\left[e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}\right]\frac{-2i}{\omega}=\frac{1}{2i}\left[e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}\right]\frac{-2i}{\omega}\times i$

The Fourier transform of $\dfrac{1}{2}\left[H(t+1/2)-H(t-1/2)\right]=\dfrac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$.

Q30. A function $y(z)$ satisfies the ordinary differential equation $y''+\dfrac{1}{z}y'-\dfrac{m^2}{z^2}y=0$, where $m=0, 1, 2, 3,.....$ Consider the four statements P, Q, R, S as given below.

P: z^m and z^{-m} are linearly independent solutions for all values of m

Q: z^m and z^{-m} are linearly independent solutions for all values of $m>0$

R: $\ln z$ and 1 are linearly independent solutions for $m=0$

S: z^m and $\ln z$ are linearly independent solutions for all values of m

The correct option for the combination of valid statements is

(a) P, R and S only (b) P and R only (c) Q and R only (d) R and S only

Ans.: (c)

Solution: $y''+\dfrac{1}{z}y'-\dfrac{m^2}{z^2}y=0\Rightarrow z^2y''+zy'-m^2y=0$, $m=0,1,2,3,...,$ $z=e^x$, $D=\dfrac{d}{dx}$

If $m=0$; $z^2y''+zy'=0$, $\left[D(D-1)+D\right]y=0 \Rightarrow \left[D^2-D+D\right]y=0$

$D^2y=0 \Rightarrow y=c_1+c_2x \Rightarrow y=c_1+c_2\ln z$ (R is correct)

And if $m\neq 0,\ m>0$, then $m\neq 0$, then $\left(D^2-m^2\right)y=0 \Rightarrow D=\pm m$

$y=c_1e^{mx}+c_2e^{-mx}=c_1e^{m\log z}+c_2e^{-m\log z}=c_1z^m+c_2z^{-m}$

or if $m\neq 0,\ m>0$, then

$y=c_1\cosh\left(m\log(z)\right)+ic_2\sinh\left(m\log(x)\right),\quad m>0$

GATE-2016

Q31. Consider the linear differential equation $\frac{dy}{dx}=xy$. If $y=2$ at $x=0$, then the value of y at $x=2$ is given by

(a) e^{-2} (b) $2e^{-2}$ (c) e^2 (d) $2e^2$

Ans.: (d)

Solution: $\frac{dy}{dx}=xy \Rightarrow \frac{1}{y}dy=xdx \Rightarrow \ln y=\frac{x^2}{2}+\ln c \Rightarrow y=ce^{x^2/2}$

If $y=2$ at $x=0 \Rightarrow c=2 \Rightarrow y=2e^{x^2/2}$.

The value of y at $x=2$ is given by $y=2e^2$

Q32. Which of the following is an analytic function of z everywhere in the complex plane?

(a) z^2 (b) $\left(z^*\right)^2$ (c) $|z|^2$ (d) $\sqrt{z}$

Ans.: (a)

Solution: $z^2=(x+iy)^2=x^2-y^2+i(2xy) \Rightarrow u=x^2-y^2$ and $v=2xy$

Cauchy Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2x,\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=2y$ satisfies.

Q33. The direction of $\vec{\nabla}f$ for a scalar field $f(x,y,z)=\frac{1}{2}x^2-xy+\frac{1}{2}z^2$ at the point $P(1,1,2)$ is

(a) $\frac{\left(-\hat{j}-2\hat{k}\right)}{\sqrt{5}}$ (b) $\frac{\left(-\hat{j}+2\hat{k}\right)}{\sqrt{5}}$ (c) $\frac{\left(\hat{j}-2\hat{k}\right)}{\sqrt{5}}$ (d) $\frac{\left(\hat{j}+2\hat{k}\right)}{\sqrt{5}}$

Ans.: (b)

Solution: $\vec{\nabla} f = (x-y)\hat{i} - x\hat{j} + z\hat{k} \Rightarrow \hat{n} = \left(\frac{\vec{\nabla} f}{|\vec{\nabla} f|}\right)_{1,1,2} = \frac{-\hat{j}+2\hat{k}}{\sqrt{5}}$

Q34. A periodic function $f(x)$ of period 2π is defined in the interval $(-\pi < x < \pi)$

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

The appropriate Fourier series expansion for $f(x)$ is

(a) $f(x) = \left(\frac{4}{\pi}\right)[\sin x + (\sin 3x)/3 + (\sin 5x)/5 + ...]$

(b) $f(x) = \left(\frac{4}{\pi}\right)[\sin x - (\sin 3x)/3 + (\sin 5x)/5 - ..]$

(c) $f(x) = \left(\frac{4}{\pi}\right)[\cos x + (\cos 3x)/3 + (\cos 5x)/5 + ...]$

(d) $f(x) = \left(\frac{4}{\pi}\right)[\cos x - (\cos 3x)/3 + (\cos 5x)/5 - ...]$

Ans.: (a)

Solution: $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

Let $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$\because \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$\Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi}(1)\,dx\right] = \frac{1}{2\pi}\left[[-x]_{-\pi}^{0} + [x]_{0}^{\pi}\right] = 0$

This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ to π is zero.

$\because \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$

$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nxdx + \int_{0}^{\pi}(1)\cos nxdx\right] = \frac{1}{\pi}\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0} + \left\{\frac{\sin nx}{n}\right\}_{0}^{\pi}\right] = 0$

$\because \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx dx + \int_{0}^{\pi}(1)\sin nx dx\right]$

$\Rightarrow b_n = \frac{1}{\pi}\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0} - \left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right] = \frac{1}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{1}{\pi}\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$

$\Rightarrow b_n = \begin{cases} 0; & n = even \\ \frac{4}{n\pi}; & n = odd \end{cases}$

Thus, Fourier series is $f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$

GATE-2017

Q35. The contour integral $\oint \frac{dz}{1+z^2}$ evaluated along a contour going from $-\infty$ to $+\infty$ along the real axis and closed in the lower half-plane circle is equal to…….. (up to two decimal places).

Ans. : π

Solution: $\oint_C \frac{1}{1+z^2}dz = \int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx + \oint_C \frac{1}{1+z^2}dz$

Poles, $1+z^2 = 0 \Rightarrow z = \pm i$, $z = -i$ is inside C

$\therefore \text{Res}(z=-i) = \lim_{z\to -i}(z+i)\frac{1}{(z-i)(z+i)} = \frac{1}{-i-i} = \frac{1}{-2i}$

$\therefore \int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx = -\frac{1}{2i}\times -2\pi i = \pi$

(Since, here we use lower half plane i.e., we traversed in clockwise direction, hence we have to take $-2\pi i$)

Q36. The coefficient of e^{ikx} in the Fourier expansion of $u(x) = A\sin^2(\alpha x)$ for $k = -2\alpha$ is

(a) $\frac{A}{4}$ (b) $\frac{-A}{4}$ (c) $\frac{A}{2}$ (d) $\frac{-A}{2}$

Ans.: (b)

Solution: Since, $\sin(\alpha x)=\dfrac{e^{i\alpha x}-e^{-i\alpha x}}{2i} \Rightarrow \sin^2(\alpha x)=\dfrac{e^{i2\alpha x}-2+e^{-2i\alpha x}}{(-4)}$

Since, $2\alpha=-k$, hence $\sin^2(\alpha x)=\dfrac{e^{-ikx}-2+e^{ikx}}{(-4)}$

Hence, $c_k=\dfrac{A}{2\pi}\int_{-\pi}^{\pi}\sin^2(\alpha x)dx=-\dfrac{A}{8\pi}\left[\int_{-\pi}^{\pi}e^{-ikx}\,e^{-ikx}dx-2\int_{-\pi}^{\pi}e^{-ikx}dx+\int_{-\pi}^{\pi}e^{-ikx}e^{ikx}dx\right]$

$$=-\frac{A}{8\pi}\left[\int_{-\pi}^{\pi}e^{-2ikx}dx-2\int_{-\pi}^{\pi}e^{-ikx}dx+\int_{-\pi}^{\pi}dx\right]$$

The first two integrals are zero and the third integral has the value 2π.

Thus,

$$c_k=-\frac{A}{8\pi}(2\pi)=-\frac{A}{4}$$

Q37. The imaginary part of an analytic complex function is $v(x,y)=2xy+3y$. The real part of the function is zero at the origin. The value of the real part of the function at $1+i$ is …….. (up to two decimal places)

Ans. : 3

Solution: The imaginary part of the given analytic function is $v(x,y)=2xy+3y$. From the Cauchy – Riemann condition

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=2x+3$$

Integrating partially gives

$$u(x,y)=x^2+3x+g(y)$$

From the second Cauchy – Riemann condition

$\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$, we obtain $\dfrac{\partial u}{\partial y}=-2y, \mu(x,y)=-y^2+g(x)$

$$\frac{dg(y)}{dy}=-2y \Rightarrow g(y)=-y^2+c$$

Hence, $u(x,y)=x^2+3x-y^2+c$

Since, the real part of the analytic function is zero at the origin.

Hence, $0=0+0-0+c \Rightarrow c=0$

Thus, $u(x,y)=x^2+3x-y^2$

$\therefore f(z)=\left(x^2+3x-y^2\right)+i(2xy+3y)$

Thus, the value of real part when

$z=1+i$, i.e. $x=1$ and $y=1$ is $u(x,y)=(1)^2+3(1)-1=3$.

Q38. Let X be a column vector of dimension $n>1$ with at least one non-zero entry. The number of non-zero eigenvalues of the matrix $M=XX^T$ is

(a) 0 (b) n (c) 1 (d) $n-1$

Ans. : (c)

Solution: Let $X=\begin{bmatrix}0\\0\\a\\0\\0\\0\end{bmatrix}$, then $X^T=\begin{bmatrix}0 & 0 & a... & 0\end{bmatrix}$

Here, X is an $n\times 1$ column vector with the entry in the i th row equal to a. X^T is a row vector having entry in the i th column equal to a. Then, XX^T is an $n\times 1$ matrix having the entry in the i th row and i th column equal to a^2.

Hence,

$$XX^T=\begin{bmatrix}0\ 0\ 0...0...0\ 0\\0\ 0\ 0...0...0\ 0\\0\ 0\ 0...0...0\ 0\\.....\\.....\\0\ 0\ 0...0...0\ 0\end{bmatrix}\rightarrow i\text{th row}$$

$$\downarrow$$

ith row

Since this matrix is diagonal, its eigenvalues are $a^2,0,0.....0$. Hence, the number of non zero eigenvalues of the matrix XX^T is 1.

Q39. Consider the differential equation $\frac{dy}{dx}+y\tan(x)=\cos(x)$. If $y(0)=0, y\left(\frac{\pi}{3}\right)$ is …….. (up to two decimal places)

Ans.: 0.52

Solution: The given differential equation is a linear differential equation of the form

$$\frac{dy}{dx} + p(x)\,y = \cos x$$

Integrating factor $= e^{\int p(x)dx}$

Thus integrating factor $= e^{\int \tan x\,dx}$

$$\Rightarrow I \cdot F = e^{\ln \sec x} = \sec x$$

Thus the general solution of the given differential equation is

$$y \cdot \sec x = \int \sec x \cdot \cos x dx + c$$

$$\Rightarrow y \sec x = x + c \qquad \text{-(i)}$$

It is given that $y(0) = 0 \Rightarrow 0 \cdot \sec 0 = 0 + c \Rightarrow c = 0$

Thus the solution satisfying the given condition is

$$y \sec x = x \Rightarrow y = \frac{x}{\sec x}$$

Thus the value of $y\left(\frac{\pi}{3}\right)$ is

$$y = \frac{\pi/3}{\sec \pi/3} = \frac{\pi/3}{2} = \frac{\pi}{6} = 0 \cdot 52$$

GATE-2018

Q40. The eigenvalues of a Hermitian matrix are all

(a) real (b) imaginary (c) of modulus one (d) real and positive

Ans. : (a)

Solution: Eigenvalue of Hermitian matrix must be real.

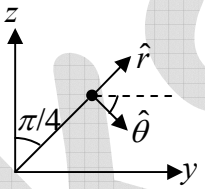
Q41. In spherical polar coordinates (r,θ,ϕ), the unit vector $\hat{\theta}$ at $(10,\pi/4,\pi/2)$ is

(a) $\hat{k}$ (b) $\frac{1}{\sqrt{2}}\left(\hat{j}+\hat{k}\right)$ (c) $\frac{1}{\sqrt{2}}\left(-\hat{j}+\hat{k}\right)$ (d) $\frac{1}{\sqrt{2}}\left(\hat{j}-\hat{k}\right)$

Ans. : (d)

Solution: $\hat{\theta}=\cos 45^0\,\hat{j}-\sin 45^0\,\hat{k}$

$$\Rightarrow \hat{\theta}=\frac{1}{\sqrt{2}}\left(\hat{j}-\hat{k}\right)$$



Q42. The scale factors corresponding to the covariant metric tensor g_{ij} in spherical polar coordinates are

(a) $1, r^2, r^2\sin^2\theta$ (b) $1, r^2, \sin^2\theta$ (c) 1,1,1 (d) $1, r, r\sin\theta$

Ans. : (d)

Q43. Given $\vec{V}_1=\hat{i}-\hat{j}$ and $\vec{V}_2=-2\hat{i}+3\hat{j}+2\hat{k}$, which one of the following $\vec{V}_3$ makes $\left(\vec{V}_1,\vec{V}_2,\vec{V}_3\right)$ a complete set for a three dimensional real linear vector space?

(a) $\vec{V}_3=\hat{i}+\hat{j}+4\hat{k}$ (b) $\vec{V}_3=2\hat{i}-\hat{j}+2\hat{k}$

(c) $\vec{V}_3=\hat{i}+2\hat{j}+6\hat{k}$ (d) $\vec{V}_3=2\hat{i}+\hat{j}+4\hat{k}$

Ans. : (d)

Solution: Let A be the matrix formed by taking $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ as column matrix i.e.,

$$A=\begin{bmatrix}V_1 & V_2 & V_3\end{bmatrix}=\begin{bmatrix}1 & -2 & 2\\ -1 & 3 & 1\\ 0 & 2 & 4\end{bmatrix}\Rightarrow |A|=-2\text{. Here } V_3=\left(2\hat{i}+\hat{j}+4\hat{k}\right)$$

Since, $|A|\neq 0$, hence, $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ form a three dimensional real vector space.

Hence, option (d) is correct.

Q44. Given

$$\frac{d^2 f(x)}{dx^2} - 2\frac{df(x)}{dx} + f(x) = 0,$$

and boundary conditions $f(0) = 1$ and $f(1) = 0$, the value of $f(0.5)$ is _____ (up to two decimal places) .

Ans. : 0.81

Solution: $\frac{d^2 f(x)}{dx^2} - 2\frac{df(x)}{dx} + f(x) = 0$

Auxiliary equation is,

$$\left(m^2 - 2m + 1\right) = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1,1$$

Hence, the solution is

$$f(x) = (c_1 + c_2 x)e^x$$

using boundary condition,

$$f(0) = c_1 e^0 \Rightarrow c_1 = 1 \qquad \text{(i)}$$

$$f(1) = (c_1 + c_2)e = 0 \qquad \text{(ii)}$$

From (i) and (ii), $c_2 = -1$

Hence, $f(x) = (1-x)e^x \Rightarrow f(0.5) = (1-0.5)e^{0.5} = 0.81$

Q45. The absolute value of the integral

$$\int \frac{5z^3 + 3z^2}{z^2 - 4} dz,$$

over the circle $|z - 1.5| = 1$ in complex plane, is _____ (up to two decimal places).

Ans. : 81.64

Solution: $f(z) = \frac{5z^3 + 3z^2}{(z-2)(z+2)}$

Pole, $z = 2, -2$

$z = -2$ is outside the center

$|-2 - 1.5| > 1$ So, will not be considered

Now, $\text{Re}\,s(2)=\lim_{z\to 2}(z-2)\dfrac{(5z^3+3z^2)}{(z-2)(z+2)}=\dfrac{52^3+32^2}{4}=\dfrac{40+12}{4}=13$

$I=2\pi i\times residue=2\pi i\times 13=26\times 3.14\Rightarrow I=81.64$

GATE-2019

Q46. For the differential equation $\dfrac{d^2y}{dx^2}-n(n+1)\dfrac{y}{x^2}=0$, where n is a constant, the product of its two independent solutions is

(a) $\dfrac{1}{x}$ (b) x (c) x^n (d) $\dfrac{1}{x^{n+1}}$

Ans. : (b)

Q47. During a rotation, vectors along the axis of rotation remain unchanged. For the rotation matrix $\begin{pmatrix}0&1&0\\0&0&-1\\-1&0&0\end{pmatrix}$, the vector along the axis of rotation is

(a) $\dfrac{1}{3}\left(2\hat{i}-\hat{j}+2\hat{k}\right)$ (b) $\dfrac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}-\hat{k}\right)$

(c) $\dfrac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}-\hat{k}\right)$ (d) $\dfrac{1}{3}\left(2\hat{i}+2\hat{j}-\hat{k}\right)$

Ans. : (b)

Q48. The pole of the function $f(z)=\cot z$ at $z=0$ is

(a) a removable pole (b) an essential singularity

(c) a simple pole (d) a second order pole

Ans. : (c)

Solution: $f(z)=\cot z$ at $z=0$

$f(z)=\dfrac{1}{\tan z}$ $z=0$ is a simple pole $f(z)=\dfrac{1}{z}\left[1-\dfrac{1}{3}z^2+....\right]$

Q49. The value of the integral $\int_{-\infty}^{\infty}\dfrac{\cos(kx)}{x^2+a^2}dx$, where $k>0$ and $a>0$, is

(a) $\dfrac{\pi}{a}e^{-ka}$ (b) $\dfrac{2\pi}{a}e^{-ka}$ (c) $\dfrac{\pi}{2a}e^{-ka}$ (d) $\dfrac{3\pi}{2a}e^{-ka}$

Ans. : (a)

Solution: $\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}dx$

$$f(z)=\frac{e^{ikx}}{z^2+a^2}=\frac{e^{ikz}}{(z+ia)(z-ia)}$$

$$I=\text{Re}.2\pi i\times\frac{e^{ik(ia)}}{2ia}=\frac{\pi e^{-ka}}{a}$$

Q50. Let θ be a variable in the range $-\pi\le\theta<\pi$. Now consider a function

$$\psi(\theta)=\begin{cases}1 & \text{for } \frac{-\pi}{2}\le\theta<\frac{\pi}{2}\\ 0 & \text{otherwise}\end{cases}$$

if its Fourier-series is written as $\psi(\theta)=\sum_m^{\infty}=-\infty C_m e^{-im\theta}$, then the value of $|C_3|^2$ (rounded off to three decimal places) is_____

CLASSICAL MECHANICS SOLUTIONS

GATE- 2010

Q1. For the set of all Lorentz transformations with velocities along the x-axis consider the two statements given below:

P: If L is a Lorentz transformation, then, L^{-1} is also a Lorentz transformation.

Q: If L_1 and L_2 are Lorentz transformations, then L_1L_2 is necessarily a Lorentz transformation.

Choose the correct option

(a) P is true and Q is false (b) Both P and Q are true

(c) Both P and Q are false (d) P is false and Q is true

Ans: (b)

Q2. A particle is placed in a region with the potential $V(x)=\frac{1}{2}kx^2-\frac{\lambda}{3}x^3$, where $k,\lambda>0$.

Then,

(a) $x=0$ and $x=\frac{k}{\lambda}$ are points of stable equilibrium

(b) $x=0$ is a point of stable equilibrium and $x=\frac{k}{\lambda}$ is a point of unstable equilibrium

(c) $x=0$ and $x=\frac{k}{\lambda}$ are points of unstable equilibrium

(d) There are no points of stable or unstable equilibrium

Ans: (b)

Solution: $V=\frac{1}{2}kx^2-\frac{\lambda x^3}{3}\Rightarrow\frac{\partial V}{\partial x}=kx-\lambda x^2=0\Rightarrow x=0,x=\frac{k}{\lambda}$.

$\Rightarrow\frac{\partial^2 V}{\partial x^2}=k-2\lambda x$

$\Rightarrow At\ x=0,\frac{\partial^2 V}{\partial x^2}=+ve$ (Stable) and at $x=\frac{k}{\lambda},\frac{\partial^2 V}{\partial x^2}=-ve$ (unstable)

Q3. A π^0 meson at rest decays into two photons, which moves along the x-axis. They are both detected simultaneously after a time, $t = 10\,s$. In an inertial frame moving with a velocity $v = 0.6c$ in the direction of one of the photons, the time interval between the two detections is

(a) $15c$ (b) $0\,s$ (c) $10\,s$ (d) $20\,s$

Ans: (a)

Solution:

$$t_1 = t_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = 10\sqrt{\frac{1+0.6}{1-0.6}} = 10\times 2 = 20\,\text{sec}\,,\; t_2 = t_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = 10\sqrt{\frac{1-0.6}{1+0.6}} = 10\times\frac{1}{2} = 5\,\text{sec}$$

$$\Rightarrow t_1 - t_2 = 15\,\text{sec}$$

Statement for Linked Answer Questions 4 and 5:

The Lagrangian for a simple pendulum is given by $L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

Q4. Hamilton's equations are then given by

(a) $\dot{p}_\theta = -mgl\sin\theta;\quad \dot{\theta} = \frac{p_\theta}{ml^2}$ (b) $\dot{p}_\theta = mgl\sin\theta;\quad \dot{\theta} = \frac{p_\theta}{ml^2}$

(c) $\dot{p}_\theta = -m\ddot{\theta};\quad \dot{\theta} = \frac{p_\theta}{m}$ (d) $\dot{p}_\theta = -\left(\frac{g}{l}\right)\theta;\quad \dot{\theta} = \frac{p_\theta}{ml}$

Ans: (a)

Solution: $H = \frac{P_\theta^2}{2ml^2} + mgl(1-\cos\theta) \Rightarrow \frac{\partial H}{\partial\theta} = -\dot{P}_\theta \Rightarrow \dot{P}_\theta = -mgl\sin\theta;\; \frac{\partial H}{\partial P_\theta} = \dot{\theta} \Rightarrow \dot{\theta} = \frac{P_\theta}{ml^2}$.

Q5. The Poisson bracket between θ and $\dot{\theta}$ is

(a) $\{\theta,\dot{\theta}\} = 1$ (b) $\{\theta,\dot{\theta}\} = \frac{1}{ml^2}$

(c) $\{\theta,\dot{\theta}\} = \frac{1}{m}$ (d) $\{\theta,\dot{\theta}\} = \frac{g}{l}$

Ans: (b)

Solution: $\{\theta,\dot{\theta}\} = \left\{\theta, \frac{P_\theta}{ml^2}\right\}$ where $\dot{\theta} = \frac{P_\theta}{ml^2} \Rightarrow \frac{1}{ml^2}\left(\frac{\partial\theta}{\partial\theta}\frac{\partial\theta}{\partial P_\theta} - \frac{\partial\theta}{\partial P_\theta}\frac{\partial P_\theta}{\partial\theta}\right) = 1\cdot\frac{1}{ml^2} - 0 = \frac{1}{ml^2}$.

GATE- 2011

Q6. A particle is moving under the action of a generalized potential $V(q,\dot{q})=\frac{1+\dot{q}}{q^2}$. The magnitude of the generalized force is

(a) $\frac{2(1+\dot{q})}{q^3}$ (b) $\frac{2(1-\dot{q})}{q^3}$ (c) $\frac{2}{q^3}$ (d) $\frac{\dot{q}}{q^3}$

Ans: (c)

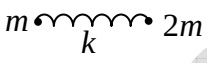
Solution: $\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}}\right)-\frac{\partial V}{\partial q}=F_q \Rightarrow F_q=\frac{2}{q^3}$

Q7. Two bodies of mass m and $2m$ are connected by a spring constant k. The frequency of the normal mode is

(a) $\sqrt{3k/2m}$ (b) $\sqrt{k/m}$ (c) $\sqrt{2k/3m}$ (d) $\sqrt{k/2m}$

Ans: (a)

Solution:



$\omega=\sqrt{\frac{k}{\mu}}=\sqrt{\frac{k}{\frac{2m}{3}}}=\sqrt{\frac{3k}{2m}}$ where reduce mass $\mu=\frac{2mm}{2m+m}=\frac{2m}{3}$.

Q8. Let (p,q) and (P,Q) be two pairs of canonical variables. The transformation

$$Q=q^{\alpha}\cos(\beta p),\ P=q^{\alpha}\sin(\beta p)$$

is canonical for

(a) $\alpha=2,\beta=\frac{1}{2}$ (b) $\alpha=2,\beta=2$ (c) $\alpha=1,\beta=1$ (d) $\alpha=\frac{1}{2},\beta=2$

Ans: (d)

Solution: $\frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\cdot\frac{\partial P}{\partial q}=1$

$$\Rightarrow \alpha q^{\alpha-1}\cos(\beta p)\times q^{\alpha}\beta\cos(\beta p)-q^{\alpha}\beta(-\sin(\beta p))\times\alpha q^{\alpha-1}\sin(\beta p)=1$$

$$\alpha q^{2\alpha-1}\beta\left(\cos^2\beta p+\sin^2\beta p\right)=1\Rightarrow \alpha\beta q^{2\alpha-1}=1\ \Rightarrow \alpha=\frac{1}{2},\beta=2.$$

Q9. Two particles each of rest mass *m* collide head-on and stick together. Before collision, the speed of each mass was 0.6 times the speed of light in free space. The mass of the final entity is

(a) $5m/4$ (b) $2m$ (c) $5m/2$ (d) $25m/8$

Ans: (c)

Solution: From conservation of energy

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}=m_1c^2 \Rightarrow \frac{2mc^2}{\sqrt{1-\frac{v^2}{c^2}}}=m_1c^2$$

Since $v=0.6c \Rightarrow m_1=5m/2$

GATE- 2012

Q10. In a central force field, the trajectory of a particle of mass *m* and angular momentum L in plane polar coordinates is given by,

$$\frac{1}{r}=\frac{m}{l^2}\left(1+\varepsilon\cos\theta\right)$$

where, ε is the eccentricity of the particle's motion. Which one of the following choice for ε gives rise to a parabolic trajectory?

(a) $\varepsilon=0$ (b) $\varepsilon=1$ (c) $0<\varepsilon<1$ (d) $\varepsilon>1$

Ans: (b)

Solution: $\frac{1}{r}=\frac{m}{l^2}\left(1+\varepsilon\cos\theta\right)$

For parabolic trajectory $\varepsilon=1$.

Q11. A particle of unit mass moves along the *x*-axis under the influence of a potential, $V(x)=x(x-2)^2$. The particle is found to be in stable equilibrium at the point $x=2$. The time period of oscillation of the particle is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

Ans: (b)

Solution: $V(x)=x(x-2)^2 \Rightarrow \frac{\partial V}{\partial x}=(x-2)^2+2x(x-2)=0 \Rightarrow x=2, x=\frac{2}{3}$

$$\frac{\partial^2 V}{\partial x^2} = 2(x-2)+2(x-2)+2x \Rightarrow \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=2} = 2\times 2 = 4$$

$$\Rightarrow \omega = \sqrt{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=2}} \Rightarrow \omega = \frac{2\pi}{T} = 2 \Rightarrow T = \pi$$

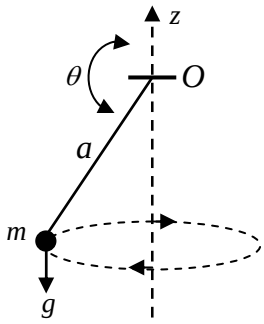
Q12. A rod of proper length l_0 oriented parallel to the *x*-axis moves with speed $2c/3$ along the x-axis in the S-frame, where c is the speed of light in free space. The observer is also moving along the x-axis with speed $c/2$ with respect to the S-frame. The length of the rod as measured by the observer is

(a) $0.35 l_0$ (b) $0.48 l_0$ (c) $0.87 l_0$ (d) $0.97 l_0$

Ans: (d)

Solution: $l = l_0\sqrt{1-\frac{u_x^2}{c^2}} = 0.97\ l_0$

Q13. A particle of mass *m* is attached to a fixed point *O* by a weightless inextensible string of length *a*. It is rotating under the gravity as shown in the figure. The Lagrangian of the particle is $L(\theta,\phi) = \frac{1}{2}ma^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) - mga\cos\theta$ where θ and ϕ are the polar angles. The Hamiltonian of the particles is



(a) $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mga\cos\theta$ (b) $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + mga\cos\theta$

(c) $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) - mga\cos\theta$ (d) $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) + mga\cos\theta$

Ans: (b)

Solution: $H = P_\theta\dot\theta + P_\phi\dot\phi - L = P_\theta\dot\theta + P_\phi\dot\phi - \frac{1}{2}ma^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) + mga\cos\theta$

$$\frac{\partial L}{\partial\dot\theta} = P_\theta \Rightarrow ma^2\dot\theta = P_\theta \Rightarrow \dot\theta = \frac{P_\theta}{ma^2} \text{ and } P_\phi = \frac{\partial L}{\partial\dot\phi} = ma^2\sin^2\theta\dot\phi \Rightarrow \dot\phi = \frac{P_\phi}{ma^2\sin^2\theta}$$

Put the value of $\dot\theta$ and $\dot\phi$

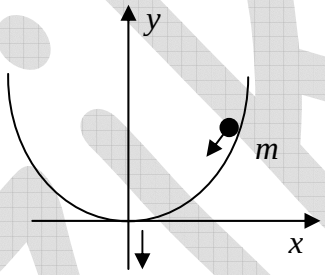
$$H = P_\theta \times \frac{P_\theta}{ma^2} + P_\phi \times \frac{P_\phi}{ma^2 \sin^2\theta} - \frac{1}{2}ma^2\left(\left(\frac{P_\theta}{ma^2}\right)^2 + \sin^2\theta\left(\frac{P_\phi}{ma^2\sin^2\theta}\right)^2\right) + mga\cos\theta$$

$$H = \frac{P_\theta^2}{ma^2} - \frac{P_\theta^2}{2ma^2} + \frac{P_\phi^2}{ma^2\sin^2\theta} - \frac{P_\phi^2}{2ma^2\sin^2\theta} + mga\cos\theta$$

$$H = \frac{1}{2ma^2}\left(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}\right) + mga\cos\theta$$

Statement for Linked Answer Questions 14 and 15:

Q14. A particle of mass m slides under the gravity without friction along the parabolic path $y = ax^2$, as shown in the figure. Here a is a constant.



The Lagrangian for this particle is given by

(a) $L = \frac{1}{2}m\dot{x}^2 - mgax^2$ (b) $L = \frac{1}{2}m\left(1+4a^2x^2\right)\dot{x}^2 - mgax^2$

(c) $L = \frac{1}{2}m\dot{x}^2 + mgax^2$ (d) $L = \frac{1}{2}m\left(1+4a^2x^2\right)\dot{x}^2 + mgax^2$

Ans: (b)

Solution: Equation of constrain is given by $y = ax^2$, $K.E., T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$

$$\dot{y} = 2ax\dot{x} \Rightarrow T = \frac{1}{2}m\left(\dot{x}^2 + 4a^2x^2\dot{x}^2\right) = \frac{1}{2}m\dot{x}^2\left(1+4ax^2\right)$$

$$V = mgy = mgax^2.$$

$$\because\ L = T - V \Rightarrow L = \frac{1}{2}m\left(1+4a^2x^2\right)\dot{x}^2 - mgax^2$$

Q15. The Lagrange's equation of motion of the particle for above question is given by

(a) $\ddot{x} = 2gax$ (b) $m\left(1+4a^2x^2\right)\ddot{x} = -2mgax - 4ma^2x\dot{x}^2$

(c) $m\left(1+4a^2x^2\right)\ddot{x} = 2mgax + 4ma^2x\dot{x}^2$ (d) $\ddot{x} = -2gax$

Ans: (b)

Solution: $\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = \frac{dL}{dx} \Rightarrow m\ddot{x}\left(1+4a^2x^2\right) = -4ma^2x\dot{x}^2 - 2mgax$

GATE- 2013

Q16. In the most general case, which one of the following quantities is NOT a second order tensor?

(a) Stress (b) Strain

(c) Moment of inertia (d) Pressure

Ans: (b)

Solution: Strain is not a tensor.

Q17. An electron is moving with a velocity of $0.85c$ in the same direction as that of a moving photon. The relative velocity of the electron with respect to photon is

(a) c (b) $-c$ (c) $0.15c$ (d) $-0.15c$

Ans: (b)

Q18. The Lagrangian of a system with one degree of freedom q is given by $L = \alpha\dot{q}^2 + \beta q^2$, where α and β are non-zero constants. If p_q denotes the canonical momentum conjugate to q then which one of the following statements is CORRECT?

(a) $p_q = 2\beta q$ and it is a conserved quantity.

(b) $p_q = 2\beta q$ and it is not a conserved quantity.

(c) $p_q = 2\alpha\dot{q}$ and it is a conserved quantity.

(d) $p_q = 2\alpha\dot{q}$ and it is not a conserved quantity.

Ans: (d)

Solution: As, $\frac{\partial L}{\partial \dot{q}} = p_q$ but $\frac{\partial L}{\partial q} \neq 0$. Thus, it is not a conserved quantity.

Q19. The relativistic form of Newton's second law of motion is

(a) $F = \dfrac{mc}{\sqrt{c^2 - v^2}}\dfrac{dv}{dt}$ (b) $F = \dfrac{m\sqrt{c^2 - v^2}}{c}\dfrac{dv}{dt}$

(c) $F = \dfrac{mc^2}{c^2 - v^2}\dfrac{dv}{dt}$ (d) $F = m\dfrac{c^2 - v^2}{c^2}\dfrac{dv}{dt}$

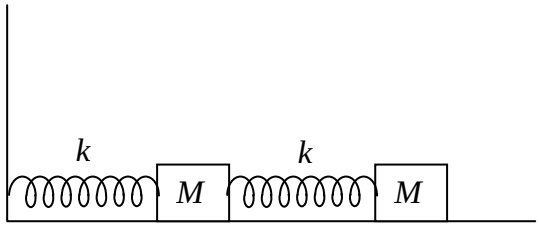
Ans: (c)

Solution: $P = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}} \Rightarrow F = \dfrac{dP}{dt} = m\dfrac{dv}{dt}\cdot\dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} + mv\left(-\dfrac{1}{2}\right)\cdot\dfrac{1}{\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}\cdot\dfrac{-2v}{c^2}\dfrac{dv}{dt}$

$$\Rightarrow F = m\frac{dv}{dt}\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}\left(1+\frac{1}{2}\frac{v^2/c^2}{\left(1-\dfrac{v^2}{c^2}\right)}\right) = m\frac{dv}{dt}\left(\frac{1-v^2/2c^2}{\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}\right)$$

$$= m\frac{dv}{dt}\left[\frac{\left(1-v^2/c^2\right)^{1/2}}{\left(1-v^2/c^2\right)\left(1-v^2/c^2\right)^{1/2}}\right] = \frac{mc^2}{\left(c^2-v^2\right)}\frac{dv}{dt}$$

Q20. Consider two small blocks, each of mass M, attached to two identical springs. One of the springs is attached to the wall, as shown in the figure. The spring constant of each spring is k. The masses slide along the surface and the friction is negligible. The frequency of one of the normal modes of the system is,



(a) $\sqrt{\dfrac{3+\sqrt{2}}{2}}\sqrt{\dfrac{k}{M}}$

(b) $\sqrt{\dfrac{3+\sqrt{3}}{2}}\sqrt{\dfrac{k}{M}}$

(c) $\sqrt{\dfrac{3+\sqrt{5}}{2}}\sqrt{\dfrac{k}{M}}$

(d) $\sqrt{\dfrac{3+\sqrt{6}}{2}}\sqrt{\dfrac{k}{M}}$

Ans: (c)

Solution: $T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2$,

$$V=\frac{1}{2}kx_1^2+\frac{1}{2}k(x_2-x_1)^2=\frac{1}{2}kx_1^2+\frac{1}{2}k\left(x_2^2+x_1^2-2x_2x_1\right)=\frac{1}{2}k\left(2x_1^2+x_2^2-2x_2x_1\right)$$

$$T=\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix};\ V=\begin{pmatrix}2k & -k\\ -k & k\end{pmatrix}$$

$$\begin{vmatrix}2k-\omega^2m & -k\\ -k & k-\omega^2m\end{vmatrix}=0\Rightarrow\left(2k-\omega^2m\right)\left(k-\omega^2m\right)-k^2=0\Rightarrow\omega=\sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$$

GATE- 2014

Q21. If the half-life of an elementary particle moving with speed 0.9c in the laboratory frame is $5\times10^{-8}s$, then the proper half-life is _____ $\times10^{-8}s.\left(c=3\times10^8 m/s\right)$

Ans: 2.18

Solution: $t=\frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$, $t_0=t\times\sqrt{1-\frac{v^2}{c^2}}=5\times10^{-8}\times\sqrt{0.19}=2.18\times10^{-8}s$

Q22. Two masses m and $3m$ are attached to the two ends of a massless spring with force constant K. If $m=100g$ and $K=0.3N/m$, then the natural angular frequency of oscillation is _____ Hz.

Ans: 0.318

Solution: $f=\frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$, $\mu=\frac{m_1.m_2}{m_1+m_2}=\frac{3m.m}{4m}=\frac{3m}{4}$, $\omega=\sqrt{\frac{4k}{3m}}=2\Rightarrow f=0.318\ Hz$

Q23. The Hamilton's canonical equation of motion in terms of Poisson Brackets are

(a) $\dot{q}=\{q,H\};\dot{p}=\{p,H\}$ (b) $\dot{q}=\{H,q\};\dot{p}=\{H,p\}$

(c) $\dot{q}=\{H,p\};\dot{p}=\{H,p\}$ (d) $\dot{q}=\{p,H\};\dot{p}=\{q,H\}$

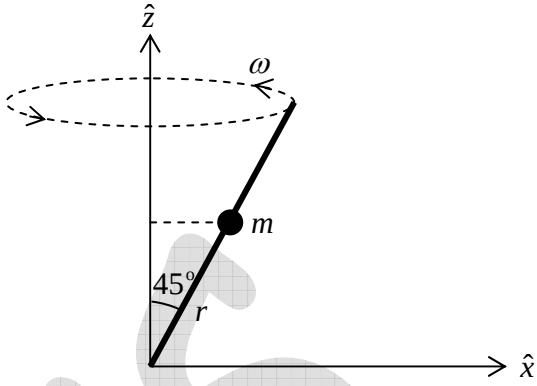
Ans: (a)

Solution: $\frac{df}{dt}=\frac{\partial f}{\partial q}.\frac{\partial q}{\partial t}+\frac{\partial f}{\partial p}.\frac{\partial p}{\partial t}+\frac{\partial f}{\partial t}$

$$\frac{df}{dt}=\frac{\partial f}{\partial q}.\frac{\partial H}{\partial p}-\frac{\partial f}{\partial p}.\frac{\partial H}{\partial q}+\frac{\partial f}{\partial t}\Rightarrow\frac{df}{dt}=\{f,H\}+\frac{\partial f}{\partial t}$$

$$\frac{dq}{dt}=\{q,H\}\ \text{ and }\ \frac{dp}{dt}=\{p,H\}$$

Q24. A bead of mass m can slide without friction along a massless rod kept at 45° with the vertical as shown in the figure. The rod is rotating about the vertical axis with a constant angular speed ω. At any instant r is the distance of the bead from the origin. The momentum conjugate to r is



(a) $m\dot{r}$

(b) $\frac{1}{\sqrt{2}}m\dot{r}$

(c) $\frac{1}{2}m\dot{r}$

(d) $\sqrt{2}m\dot{r}$

Ans: (a)

Solution: $L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2)-mgr\cos\theta$

Equation of constrain is $\theta=\frac{\pi}{4}$ and it is given $\dot{\phi}=\omega$

$$L=\frac{1}{2}m(\dot{r}^2+\frac{1}{2}r^2\omega^2)-\frac{1}{\sqrt{2}}mgr$$

Thus the momentum conjugate to r is $p_r=\frac{\partial L}{\partial \dot{r}} \Rightarrow p_r=m\dot{r}$

Q25. A particle of mass m is in a potential given by

$$V(r)=-\frac{a}{r}+\frac{ar_0^2}{3r^3}$$

where a and r_0 are positive constants. When disturbed slightly from its stable equilibrium position it undergoes a simple harmonic oscillation. The time period of oscillation is

(a) $2\pi\sqrt{\frac{mr_0^3}{2a}}$ (b) $2\pi\sqrt{\frac{mr_0^3}{a}}$ (c) $2\pi\sqrt{\frac{2mr_0^3}{a}}$ (d) $4\pi\sqrt{\frac{mr_0^3}{a}}$

Ans: (a)

Solution: $V(r)=-\frac{a}{r}+\frac{ar_0^2}{3r^3}$,

For equilibrium $\frac{\partial V}{\partial r}=\frac{a}{r^2}-\frac{3ar_0^2}{3r^4}=0, \quad r=\pm r_0$

$$\frac{\partial^2 V}{\partial r^2}=-\frac{2a}{r^3}+\frac{4ar_0^2}{r^5}\bigg|_{r_0}=-\frac{2a}{r_0^3}+\frac{4ar_0^2}{r_0^5}=\frac{2a}{r_0^3}$$

$$\omega = \sqrt{\frac{\left.\frac{\partial^2 V}{\partial r^2}\right|_{r_0}}{m}} \Rightarrow T = 2\pi\sqrt{\frac{mr_0^3}{2a}}$$

Q26. A planet of mass m moves in a circular orbit of radius r_0 in the gravitational potential $V(r) = -\frac{k}{r}$, where k is a positive constant. The orbit angular momentum of the planet is

(a) $2r_0 km$ (b) $\sqrt{2r_0 km}$ (c) $r_0 km$ (d) $\sqrt{r_0 km}$

Ans: (d)

Solution: $V_{effctive} = \frac{J^2}{2mr^2} - \frac{k}{r} \Rightarrow \frac{dV_{effect}}{dr} = -\frac{J^2}{mr^3} + \frac{k}{r^2} = 0$ at $r = r_0$

so $J = \sqrt{r_0 km}$

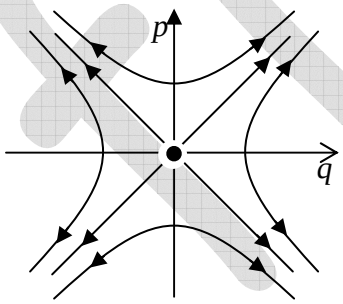
Q27. Given that the linear transformation of a generalized coordinate q and the corresponding momentum p, $Q = q + 4ap$, $P = q + 2p$ is canonical, the value of the constant a is ____

Ans: 0.25

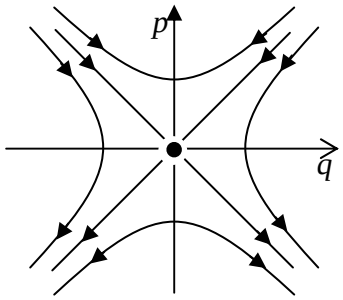
Solution: $\frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} = 1 \Rightarrow 1\times 2 - 4a\times 1 = 1 \Rightarrow a = 0.25$

Q28. The Hamiltonian of particle of mass m is given by $H = \frac{p^2}{2m} - \frac{\alpha q^2}{2}$. Which one of the following figure describes the motion of the particle in phase space?

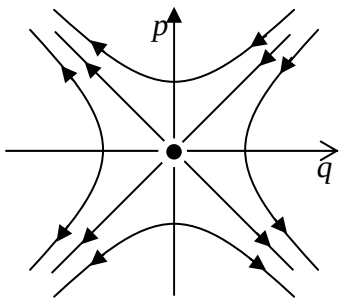
(a)



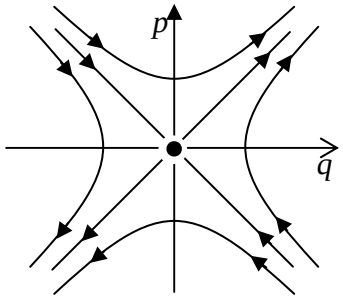
(b)



(c)



(d)



Ans: (d)

GATE- 2015

Q29. A satellite is moving in a circular orbit around the Earth. If T, V and E are its average kinetic, average potential and total energies, respectively, then which one of the following options is correct?

(a) $V = -2T; E = -T$ (b) $V = -T; E = 0$

(c) $V = -\frac{T}{2}; E = \frac{T}{2}$ (d) $V = \frac{-3T}{2}; E = \frac{-T}{2}$

Ans.: (a)

Solution: From Virial theorem $\langle T \rangle = \frac{n+1}{2}\langle V \rangle$ where $V \propto r^{n+1}$

$$\because V = \frac{-k}{r} \Rightarrow V \propto \frac{1}{r} \Rightarrow n = -2 \Rightarrow \langle V \rangle = -2\langle T \rangle$$

Q30. In an inertial frame S, two events A and B take place at $(ct_A = 0, \vec{r}_A = 0)$ and $(ct_B = 0, \vec{r}_B = 2\hat{y})$, respectively. The times at which these events take place in a frame S' moving with a velocity $0.6c\,\hat{y}$ with respect to S are given by

(a) $ct'_A = 0;\ ct'_B = -\frac{3}{2}$ (b) $ct'_A = 0;\ ct'_B = 0$

(c) $ct'_A = 0;\ ct'_B = \frac{3}{2}$ (d) $ct'_A = 0;\ ct'_B = \frac{1}{2}$

Ans.: (a)

Solution: Velocity of S' with respect to S is $v = 0.6c$

$$t'_A = \frac{t_A - \frac{v}{c^2}y}{\sqrt{1-\frac{v^2}{c^2}}}$$

For event A, $t_A = 0, y = 0$. So $ct'_A = 0$

$$t'_B = \frac{t_B - \frac{v}{c^2}y}{\sqrt{1-\frac{v^2}{c^2}}}$$

For event B, $t_B = 0, y = 2$. So $ct_B^{'} = -\frac{3}{2}$

Q31. The Lagrangian for a particle of mass m at a position $\vec{r}$ moving with a velocity $\vec{v}$ is given by $L = \frac{m}{2}\vec{v}^2 + C\vec{r}.\vec{v} - V(r)$, where $V(r)$ is a potential and C is a constant. If $\vec{p}_c$ is the canonical momentum, then its Hamiltonian is given by

(a) $\frac{1}{2m}(\vec{p}_c + C\vec{r})^2 + V(r)$ (b) $\frac{1}{2m}(\vec{p}_c - C\vec{r})^2 + V(r)$

(c) $\frac{p_c^2}{2m} + V(r)$ (d) $\frac{1}{2m}p_c^2 + C^2r^2 + V(r)$

Ans.: (b)

Solution: $L = \frac{m}{2}\vec{v}^2 + C\vec{r}.\vec{v} - V(r)$ where $v = \dot{r}$

$$H = \sum \dot{r}p_c - L = \dot{r}p_c - L$$

$$\Rightarrow \frac{\partial L}{\partial \dot{r}} = p_c = (m\dot{r} + Cr) \Rightarrow \dot{r} = \frac{p_c - Cr}{m}$$

$$\Rightarrow H = \left(\frac{p_c - Cr}{m}\right)p_c - \frac{m}{2}\left(\frac{p_c - Cr}{m}\right)^2 - cr\left(\frac{p_c - Cr}{m}\right) + V(r)$$

$$\Rightarrow H = \left(\frac{p_c - Cr}{m}\right)(p_c - Cr) - \frac{m}{2}\left(\frac{p_c - Cr}{m}\right)^2 + V(r)$$

$$\Rightarrow H = \frac{(p_c - Cr)^2}{m} - \frac{(p_c - Cr)^2}{2m} + V(r) \quad \Rightarrow H = \frac{1}{2m}(p_c - Cr)^2 + V(r)$$

Q32. The Hamiltonian for a system of two particles of masses m_1 and m_2 at $\vec{r}_1$ and $\vec{r}_2$ having velocities $\vec{v}_1$ and $\vec{v}_2$ is given by $H = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{C}{(\vec{r}_1 - \vec{r}_2)^2}\hat{z}\cdot(\vec{r}_1 \times \vec{r}_2)$, where C is constant. Which one of the following statements is correct?

(a) The total energy and total momentum are conserved

(b) Only the total energy is conserved

(c) The total energy and the z - component of the total angular momentum are conserved

(d) The total energy and total angular momentum are conserved

Ans.: (c)

Solution: Lagrangian is not a function of time, so energy is conserved and component of $(\vec{r}_1 \times \vec{r}_2)$ is only in $\hat{z}$ direction means potential is symmetric under ϕ, so L_z is conserved.

Q33. A particle of mass $0.01\,kg$ falls freely in the earth's gravitational field with an initial velocity $(0) = 10 ms^{-1}$. If the air exerts a frictional force of the form, $f = -kv$, then for $k = 0.05\,Nm^{-1}s$, the velocity (in ms^{-1}) at time $t = 0.2\,s$ is _____ (upto two decimal places). (use $g = 10\,ms^{-2}$ and $e = 2.72$)

Ans.: 4.94

Solution: $m\frac{dv}{dt} = mg - kv \Rightarrow \frac{dv}{dt} = g - \frac{k}{m}v \Rightarrow \frac{dv}{g - \frac{k}{m}v} = dt \Rightarrow \int_{10}^{u} \frac{dv}{g - \frac{k}{m}v} = \int_{0}^{0.2} dt$

$$\Rightarrow -\frac{m}{k}\left[\ln\left[g - \frac{k}{m}v\right]\right]_{10}^{u} = [t]_0^{0.2} \Rightarrow -\frac{m}{k}\left\{\left[\ln\left(g - \frac{k}{m}u\right)\right] - \ln\left(g - \frac{10k}{m}\right)\right\} = 0.2$$

$$\Rightarrow -\frac{m}{k}\left\{\ln\left(10 - \frac{0.05}{0.01}u\right) - \ln\left(10 - 10 \times \frac{.05}{.01}\right)\right\} = 0.2$$

$$\Rightarrow -\frac{m}{k}\{\ln(10 - 5u) - \ln(-40)\} = 0.2$$

$$\ln\left(\frac{8}{u-2}\right) = \frac{0.2k}{m} \Rightarrow \ln\left(\frac{8}{u-2}\right) = \frac{0.2k}{m} \Rightarrow \frac{8}{u-2} = e$$

$$\Rightarrow u = \frac{8}{e} + 2 = 4.94\ m/s$$

Q34. Consider the motion of the Sun with respect to the rotation of the Earth about its axis. If $\vec{F}_c$ and $\vec{F}_{Co}$ denote the centrifugal and the Coriolis forces, respectively, acting on the Sun, then

(a) $\vec{F}_c$ is radially outward and $\vec{F}_{Co} = \vec{F}_c$ (b) $\vec{F}_c$ is radially inward and $\vec{F}_{Co} = -2\vec{F}_c$

(c) $\vec{F}_c$ is radially outward and $\vec{F}_{Co} = -2\vec{F}_c$ (d) $\vec{F}_c$ is radially outward and $\vec{F}_{Co} = 2\vec{F}_c$

Ans.: (b)

Q35. A particle with rest mass M is at rest and decays into two particles of equal rest masses $\frac{3}{10}M$ which move along the z axis. Their velocities are given by

(a) $\vec{v}_1 = \vec{v}_2 = (0.8c)\hat{z}$ (b) $\vec{v}_1 = -\vec{v}_2 = (0.8c)\hat{z}$

(c) $\vec{v}_1 = -\vec{v}_2 = (0.6c)\hat{z}$ (d) $\vec{v}_1 = (0.6c)\hat{z};\ \vec{v}_2 = (-0.8c)\hat{z}$

Ans.: (b)

Solution: $M \to \frac{3}{10}M + \frac{3}{10}M$

From momentum conservation

$$0 = \vec{P}_1 + \vec{P}_2 \Rightarrow \vec{P}_1 = -\vec{P}_2 \Rightarrow |P_1| = |P_2|$$

From energy conservation

$$E = E_1 + E_2$$

$$\Rightarrow Mc^2 = \frac{3}{10}\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{3}{10}\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow Mc^2 = \frac{3}{5}\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\left(1-\frac{v^2}{c^2}\right) = \frac{9}{25} \Rightarrow \frac{v^2}{c^2} = \frac{16}{25} \Rightarrow v = 0.8c$$

GATE-2016

Q36. The kinetic energy of a particle of rest mass m_0 is equal to its rest mass energy. Its momentum in units of $m_0 c$, where c is the speed of light in vacuum, is _____.

(Give your answer upto two decimal places)

Ans. : 1.73

Solution: $m_0c^2 = E - m_0c^2 \Rightarrow E = 2m_0c^2$

$$\Rightarrow \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2m_0c^2 \Rightarrow v = \frac{\sqrt{3}}{2}c$$

$$\because E^2 = p^2c^2 + m_0^2c^4 \Rightarrow 4m_0^2c^4 - m_0^2c^4 = p^2c^2 \Rightarrow p = \sqrt{3}m_0c = 1.732m_0c$$

Q37. In an inertial frame of reference S, an observer finds two events occurring at the same time at coordinates $x_1 = 0$ and $x_2 = d$. A different inertial frame S' moves with velocity v with respect to S along the positive x-axis. An observer in S' also notices these two

events and finds them to occur at times t_1' and t_2' and at positions x_1' and x_2' respectively.

If $\Delta t' = t_2' - t_1', \Delta x' = x_2' - x_1'$ and $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$, which of the following statements is true?

(a) $\Delta t' = 0, \Delta x' = \gamma d$ (b) $\Delta t' = 0, \Delta x' = \dfrac{d}{\gamma}$

(c) $\Delta t' = \dfrac{-\gamma v d}{c^2}, \Delta x' = \gamma d$ (d) $\Delta t' = \dfrac{-\gamma v d}{c^2}, \Delta x' = \dfrac{d}{\gamma}$

Ans.: (c)

Solution: $t_2' - t_1' = \left(\dfrac{t_2 - \dfrac{v x_2}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) - \left(\dfrac{t_1 - \dfrac{v x_1}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) \Rightarrow \Delta t' = \gamma \Delta t - \dfrac{\gamma v \Delta x}{c^2}$

It is given, $\Delta t = 0, \Delta x = d$

$\Rightarrow \Delta t' = -\dfrac{\gamma v \Delta x}{c^2} = -\dfrac{\gamma v d}{c^2}$

$x_2' - x_1' = \left(\dfrac{x_2 - v t_2}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) - \left(\dfrac{x_1 - v t_1}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) \Rightarrow \Delta x' = \gamma\left(\Delta x - v \Delta t\right)$

$\Rightarrow \Delta x' = \gamma d$.

Q38. The Lagrangian of a system is given by

$L = \dfrac{1}{2} m l^2 \left[\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right] - mgl\cos\theta$, where m, l and g are constants.

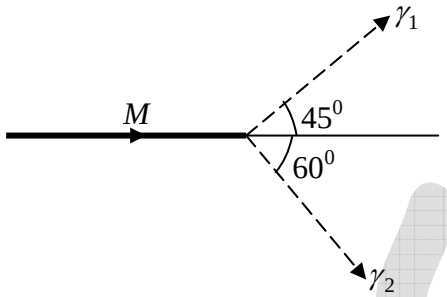
Which of the following is conserved?

(a) $\dot{\varphi}\sin^2\theta$ (b) $\dot{\varphi}\sin\theta$ (c) $\dfrac{\dot{\varphi}}{\sin\theta}$ (d) $\dfrac{\dot{\varphi}}{\sin^2\theta}$

Ans.: (a)

Solution: As φ is cyclic coordinate, so $\dfrac{\partial L}{\partial \dot{\varphi}} = p_\varphi = m l^2 \sin^2\dot{\varphi}$, is a constant since m, l and g are constants. Thus $\dot{\varphi}\sin^2\theta$ is conserved.

Q39. A particle of rest mass M is moving along the positive x-direction. It decays into two photons γ_1 and γ_2 as shown in the figure. The energy of γ_1 is $1\,GeV$ and the energy of γ_2 is $0.82\,GeV$. The value of M (in units of $\frac{GeV}{c^2}$) is _____. (Give your answer upto two decimal places)



Ans.: 1.44

Solution: $\sqrt{p^2c^2+M^2c^4}=E_1+E_2=1.82GeV$

$$p=\frac{E_1}{c}\cos\theta_1+\frac{E_2}{c}\cos\theta_2=\frac{1GeV}{c}\frac{1}{\sqrt{2}}+\frac{0.82GeV}{c}\frac{1}{2}=\frac{1.11GeV}{c}$$

$$\Rightarrow p^2c^2+m^2c^4=3.312\Rightarrow m^2c^4=3.312-1.23=2.08$$

$$\Rightarrow m=\sqrt{2.076}=1.44$$

GATE- 2017

Q40. If the Lagrangian $L_0=\frac{1}{2}m\left(\frac{dq}{dt}\right)^2-\frac{1}{2}m\omega^2q^2$ is modified to $L=L_0+\alpha q\left(\frac{dq}{dt}\right)$, which one of the following is TRUE?

(a) Both the canonical momentum and equation of motion do not change

(b) Canonical momentum changes, equation of motion does not change

(c) Canonical momentum does not change, equation of motion changes

(d) Both the canonical momentum and equation of motion change

Ans. : (b)

Solution: For Lagrangian $L_0=\frac{1}{2}m\left(\frac{dq}{dt}\right)^2-\frac{1}{2}m\omega^2q^2$ canonical momentum is $p=m\dot{q}$ and equation of motion is given by $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=0\Rightarrow m\ddot{q}+m\omega^2q=0$

For Lagrangian $L = L_0 + \alpha q\left(\frac{dq}{dt}\right) \Rightarrow L = \frac{1}{2}m\left(\frac{dq}{dt}\right)^2 - \frac{1}{2}m\omega^2 q^2 + \alpha q\dot{q}$

Canonical momentum is $p = m\dot{q} + \alpha q$

Equation of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0 \Rightarrow m\ddot{q} + m\omega^2 q = 0$$

Q41. Two identical masses of $10\,gm$ each are connected by a massless spring of spring constant $1\,N/m$. The non-zero angular eigenfrequency of the system is……rad/s. (up to two decimal places)

Ans. : 14.14

Solution: $\omega = \sqrt{\frac{k}{\mu}}$, where $\mu = \frac{m}{2} = \frac{10}{2\times 1000} = \frac{1}{200}$ and $k = 1N/m$, $\omega = 14.14$

Q42. The phase space trajectory of an otherwise free particle bouncing between two hard walls elastically in one dimension is a

(a) straight line (b) parabola (c) rectangle (d) circle

Ans. : (c)

Solution: $E = \frac{p^2}{2m}$, $p = \pm\sqrt{2mE}$

Q43. The Poisson bracket $\left[x, xp_y + yp_x\right]$ is equal to

(a) $-x$ (b) y (c) $2p_x$ (d) p_y

Ans. : (b)

Solution: $\left[x, xp_y + yp_x\right] = \left[x, xp_y\right] + \left[x, yp_x\right] = 0 + y\left[x, p_x\right] = y$

Q44. An object travels along the x-direction with velocity $\frac{c}{2}$ in a frame O. An observer in a frame O' sees the same object travelling with velocity $\frac{c}{4}$. The relative velocity of O' with respect to O in units of c is…….. (up to two decimal places).

Ans. : 0.28

Solution: $u_x^{'}=\frac{c}{2}, v=\frac{c}{4}$, $u_x=\frac{u_x^{'}-v}{1-\frac{u_x^{'}v}{c^2}}=\frac{\frac{c}{2}-\frac{c}{4}}{1-\frac{c}{2}.\frac{c}{4}.\frac{1}{c^2}}=\frac{2c}{7}=0.28c$

Q45. A uniform solid cylinder is released on a horizontal surface with speed $5\,m/s$ without any rotation (slipping without rolling). The cylinder eventually starts rolling without slipping. If the mass and radius of the cylinder are $10\,gm$ and $1cm$ respectively, the final linear velocity of the cylinder is…… m/s. (up to two decimal places).

Ans. : 3.33

Solution: $mvr=mv_{cm}r+I_{cm}\omega=mv_{cm}r+\frac{1}{2}mr^2\frac{v_{cm}}{r}\Rightarrow v=\frac{3}{2}v_{cm}\Rightarrow v_{cm}=\frac{2}{3}v=\frac{10}{3}=3.33m/\sec$

Q46. A person weighs w_p at Earth's north pole and w_e at the equator. Treating the Earth as a perfect sphere of radius 6400 km, the value $100\times\frac{\left(w_p-w_e\right)}{w_p}$ is…….. (up to two decimal places). (Take $g=10\,ms^{-2}$).

Ans. : 0.33

Solution: $g_p=g$, $g_e=g-\omega^2R\Rightarrow 100\times\frac{w_p-w_e}{w_p}=\frac{\omega^2R}{g}$

Now, $g=10\,m/\sec^2$ and $R=6400\times10^3m$

$\Rightarrow\omega=\frac{2\pi}{T}=\frac{2\pi}{24\times3600}$

Then $100\times\frac{w_p-w_e}{w_p}=0.33$

GATE - 2018

Q47. In the context of small oscillations, which one of the following does NOT apply to the normal coordinates?

(a) Each normal coordinate has an eigen-frequency associated with it

(b) The normal coordinates are orthogonal to one another

(c) The normal coordinates are all independent

(d) The potential energy of the system is a sum of squares of the normal coordinates with constant coefficients

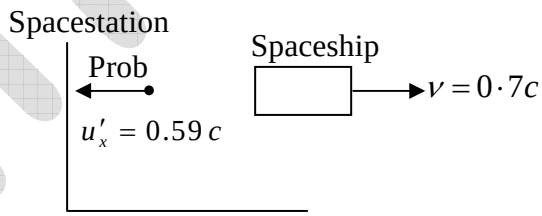
Ans. : (b)

Solution: Normal co-ordinate must be independent. It is not necessary that it should orthogonal.

Q48. A spaceship is travelling with a velocity of $0.7c$ away from a space station. The spaceship ejects a probe with a velocity $0.59c$ opposite to its own velocity. A person in the space station would see the probe moving at a speed Xc, where the value of X is _____ (up to three decimal places).

Ans.: $0.187c$

Solution: $v = 0\cdot 7c$, $u_x' = -0\cdot 59c$,



$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c_2}}$$

$$u_x = \frac{-0\cdot 59c + 0\cdot 7c}{1 - 0\cdot 7 \times 0\cdot 59} = \frac{0.11c}{1 - 0.413} = \frac{0.11c}{0.587} = 0.187\,c$$

Q49. An interstellar object has speed v at the point of its shortest distance R from a star of much larger mass M. Given $v^2 = 2GM/R$, the trajectory of the object is

(a) circle (b) ellipse (c) parabola (d) hyperbola

Ans. : (c)

Solution: At shortest distance $E = \frac{J^2}{2mR^2} - \frac{GMm}{R}$

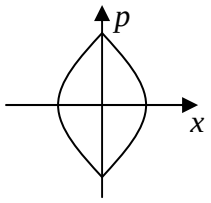
Since, $mvR = J \Rightarrow J^2 = m^2 v^2 R^2$

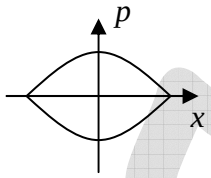
Now, $J^2 = m^2 2GMR = 2GMm^2 R$ (Given that $v^2 = \frac{2GM}{R}$)

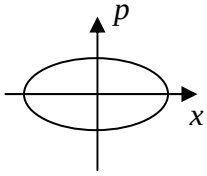
$$E=\frac{2GMm^2R}{2mR^2}-\frac{GMm}{R}=\frac{GMm}{R}-\frac{GMm}{R}=0$$

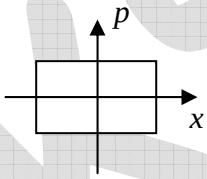
For Kepler's potential, if energy is zero, then the shape is parabola.

Q50. A particle moves in one dimension under a potential $V(x)=\alpha|x|$ with some non-zero total energy. Which one of the following best describes the particle trajectory in the phase space?

(a) 

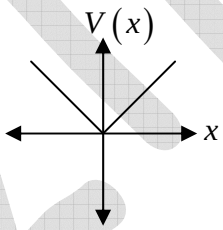
(b) 

(c) 

(d) 

Ans.: (a)

Solution: $E=\frac{p^2}{2m}+\alpha|x|$

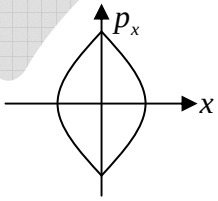


For $x>0$, $E=\frac{p^2}{2m}+\alpha x$

$\Rightarrow p^2=2m(E-\alpha x)$

For $x<0$, $E=\frac{p^2}{2m}-\alpha x$

$\Rightarrow p^2=2m(E+\alpha x)$



Q51. If H is the Hamiltonian for a free particle with mass m, the commutator $\left[x,\left[x,H\right]\right]$ is

(a) $\hbar^2/m$ (b) $-\hbar^2/m$ (c) $-\hbar^2/(2m)$ (d) $\hbar^2/(2m)$

Ans. : (b)

Solution: For free particle, potential is zero.

$$\Rightarrow H=\frac{P_x^2}{2m}$$

Now, $\left[x,H\right]=\left[x,\frac{P_x^2}{2m}\right]=\frac{2i\hbar}{2m}P_x$

$\left[x,\left[x,H\right]\right]=\frac{2i\hbar}{2m}\left[x,P_x\right]=\frac{i\hbar}{m}\left(i\hbar\right)=-\frac{\hbar^2}{m}$

Q52. For the transformation

$$Q=\sqrt{2q}\,e^{-1+2\alpha}\cos p, P=\sqrt{2q}\,e^{-\alpha-1}\sin p$$

(where α is a constant) to be canonical, the value of α is _____.

Ans. : 2

Solution: $Q=\sqrt{2q}e^{-1+2\alpha}\cos p, P=\sqrt{2q}e^{-\alpha-1}.\sin p$

Since, $\left[Q,P\right]=1$

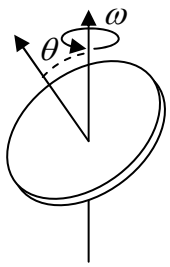
$\Rightarrow\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}=1$

$\Rightarrow\left(\frac{1}{2}\sqrt{2}q^{-\frac{1}{2}}.e^{-1+2\alpha}\cos p\right)\left(\sqrt{2q}e^{-\alpha-1}\cos p\right)-\sqrt{2q}e^{-1+2\alpha}\left(-\sin p\right).\frac{\sqrt{2}}{2}q^{-\frac{1}{2}}e^{-\alpha.1}\sin p=1$

$\Rightarrow e^{\alpha-2}.\left[\cos^2 p+\sin^2 p\right]=1=e^0$

$\Rightarrow\alpha=2$

Q53. A uniform circular disc of mass m and radius R is rotating with angular speed ω about an axis passing through its centre and making an angle $\theta=30^0$ with the axis of the disc. If the kinetic energy of the disc is $\alpha m\omega^2R^2$, the value of α is_____ (up to two decimal places).



Ans. : 0.21

Solution: The kinetic energy of the disc is,

$T=\frac{1}{2}\vec{L}\cdot\vec{\omega}$

Where $\vec{L}$ is angular momentum and ω is angular velocity

$T=\frac{1}{2}\left|\vec{L}\right|\left|\vec{\omega}\right|\cos 30^0=\frac{1}{2}I\omega\cdot\omega\frac{\sqrt{3}}{2}=\frac{1}{2}\left(\frac{mR^2}{2}\right)\omega^2\times\frac{\sqrt{3}}{2}$

$T=\frac{\sqrt{3}}{8}m\omega^2R^2=0.21m\omega^2R^2\Rightarrow\alpha m\omega^2R^2=0.21m\omega^2R^2$

Hence, $\alpha=0.21$

GATE-2019

Q54. Consider a transformation from one set of generalized coordinate and momentum (q, p) to another set (Q, P) denoted by,

$$Q = pq^s; \quad P = q^r$$

where s and r are constants. The transformation is canonical if

(a) $s = 0$ and $r = 1$ (b) $s = 2$ and $r = -1$

(c) $s = 0$ and $r = -1$ (d) $s = 2$ and $r = 1$

Ans. : (b)

Solution: $\frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} = 1 \Rightarrow 0 - q^s r q^{r-1} = 1$

$-rq^{r+s-1} = 1 \Rightarrow \ s = 2$ and $r = -1$

Q55. The Hamiltonian for a particle of mass m is $H = \frac{p^2}{2m} + kqt$ where q and p are the generalized coordinate and momentum, respectively, t is time and k is a constant. For the initial condition, $q = 0$ and $p = 0$ at $t = 0, q(t) \propto t^\alpha$. The value of α is _____

Ans. : 3

Solution: $\frac{\partial H}{\partial p} = \dot{q} = \frac{p}{m}$ ....(1)

$\frac{\partial H}{\partial q} = -\dot{p} = kt \Rightarrow p = -\frac{kt^2}{2}$ ….(2)

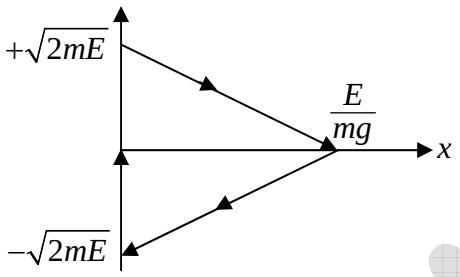
$\frac{dq}{dt} = -\frac{kt^2}{2} \Rightarrow q = -\frac{kt^3}{6} \ q \propto t^3$ so $\alpha = 3$

Q56. A ball bouncing of a rigid floor is described by the potential energy function

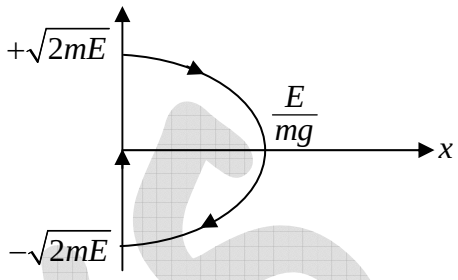
$$V(x)=\begin{cases} mgx & \text{for} \quad x>0 \\ \infty & \text{for} \quad x\le 0 \end{cases}$$

Which of the following schematic diagrams best represents the phase space plot of the ball?

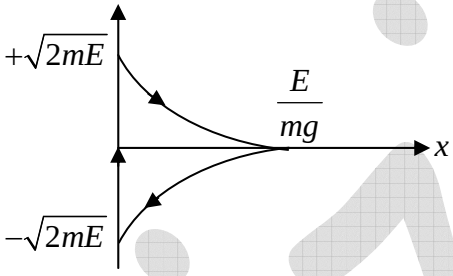
(a)



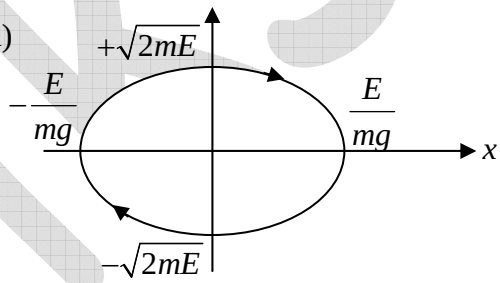
(b)



(c)



(d)



Ans. : (b)

Solution: $E=\dfrac{p^2}{2m}+mgx \Rightarrow p^2=2m\left(E-mgx\right)$ which is equation of parabola

Q57. Consider the Hamiltonian $H\left(q,p\right)=\dfrac{ap^2q^4}{2}+\dfrac{\beta}{q^2}$, where α and β are parameters with appropriate dimensions, and q and p are the generalized coordinate and momentum, respectively. The corresponding Lagrangian $L\left(q,\dot{q}\right)$ is

(a) $\dfrac{1}{2\alpha}\dfrac{\dot{q}^2}{q^4}-\dfrac{\beta}{q^2}$ (b) $\dfrac{1}{2\alpha}\dfrac{\dot{q}^2}{q^4}+\dfrac{\beta}{q^2}$ (c) $\dfrac{1}{\alpha}\dfrac{\dot{q}^2}{q^4}+\dfrac{\beta}{q^2}$ (d) $-\dfrac{1}{2\alpha}\dfrac{\dot{q}^2}{q^4}+\dfrac{\beta}{q^2}$

Ans. : (a)

Solution: $L=p\dot{q}-H \Rightarrow p\dot{q}-\dfrac{ap^2q^4}{2}-\dfrac{\beta}{q^2}$ from Hamiltonian equation of motion

$$\frac{\partial H}{\partial p}=\dot{q} \Rightarrow p=\frac{\dot{q}}{aq^4}$$

$$L=\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4}-\frac{\beta}{q^2}$$

Q58. A projectile of mass $1kg$ is launched at an angle of 30^0 from the horizontal direction at $t=0$ and takes time T before hitting the ground. If its initial speed is 10 ms^{-1}, the value of the action integral for the entire flight in the units of kgm^2s^{-1} (round off to one decimal place) is_____. [Take $g=10\,ms^{-2}$]

Ans. : 33.3

Solution: $T=\frac{2v\sin\theta}{g}=1\sec$

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-mgy$$

$$\dot{x}=v\cos\theta=5\sqrt{3}ms^{-1}\quad \dot{y}=v\sin\theta-gt=5-10t$$

$$y=ut-\frac{1}{2}gt^2=v\sin\theta t-\frac{1}{2}gt^2=10.\frac{1}{2}t-\frac{1}{2}10t^2=5t-5t^2$$

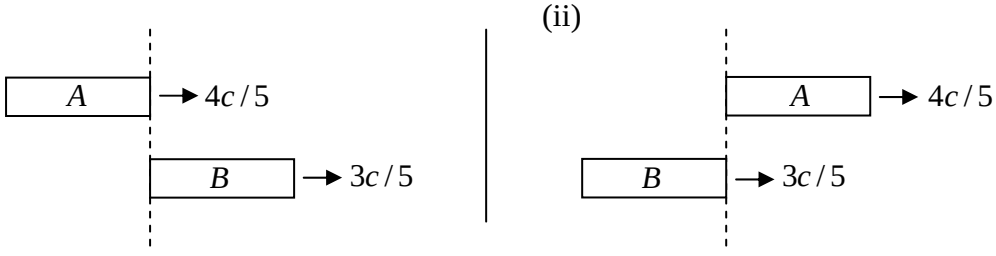
$$L=\frac{1}{2}1\left(\left(5\sqrt{3}\right)^2+\left(5-10t\right)^2\right)-1\times10\times\left(5t-5t^2\right)$$

$$L=100t^2-100t+50$$

$$A=\int_0^T Ldt=\int_0^1\left(100t^2-100t+50\right)dt=33.3$$

Q59. Two spaceships A and B, each of the same rest length L, are moving in the same direction with speeds $\frac{4c}{5}$ and $\frac{3c}{5}$, respectively, where c is the speed of light. As measured by B, the time taken by A to completely overtake B [see figure below] in units of L/c (to the nearest integer) is _____

(i) (ii)



Ans. : 5

Solution: $u_{A,B}=\dfrac{\frac{4}{5}c-\frac{3}{5}c}{1-\frac{4}{5}c.\frac{3}{5}c.\frac{1}{c^2}}=\dfrac{\frac{c}{5}}{\frac{13}{25}}=\dfrac{5}{13}c$

Kinematic equation is given by

$$\frac{5}{13}c\times t=L\sqrt{1-\frac{25}{169}}+L\Rightarrow t=\frac{5L}{c}\Rightarrow\alpha=5$$

Q60. Two events, one on the earth and the other one on the Sun, occur simultaneously in the earth's frame. The time difference between the two events as seen by an observer in a spaceship moving with velocity $0.5c$ in the earth's frame along the line joining the earth to the Sun is Δt, where c is the speed of light. Given that light travels from the Sun to the earth in 8.3 minutes in the earth's frame, the value of $|\Delta t|$ in minutes (rounded off to two decimal places) is_____

(Take the earth's frame to be inertial and neglect the relative motion between the earth and the sun)

Ans. : 4.77

Solution: $t_2^{'}-t_1^{'}=0 \quad x_2^{'}-x_1^{'}=8.3\times3\times10^8\times60 \quad v=0.5c$

$$\Delta t=t_2-t_1=\left(\frac{t_2^{'}+\frac{vx_2^{'}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)-\left(\frac{t_1^{'}+\frac{vx_1^{'}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\left(\frac{t_2^{'}-t_1^{'}}{\sqrt{1-\frac{v^2}{c^2}}}\right)+\frac{v}{c^2}\frac{\left(x_2^{'}-x_1^{'}\right)}{\sqrt{1-\frac{v^2}{c^2}}}=4.77\,\text{min}$$

ELECTROMAGNETIC THEORY SOLUTIONS

GATE- 2010

Q1. An insulating sphere of radius a carries a charge density

$$\rho(\vec{r}) = \rho_0(a^2 - r^2)\cos\theta; r < a.$$

The leading order term for the electric field at a distance d, far away from the charge distribution, is proportional to

(a) d^{-1} (b) d^{-2} (c) d^{-3} (d) d^{-4}

Ans: (c)

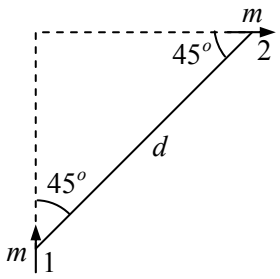
Solution: $V(r) = \left[\frac{1}{r}\int_V \rho d\tau + \frac{1}{r^2}\int \rho\cos\theta d\tau + \cdots\right]$,

Ist term, $\int \rho d\tau = \int_0^a\int_0^\pi\int_0^{2\pi} \rho_0(a^2 - r^2)\cos\theta \times r^2\sin\theta dr d\theta d\phi = 0$

IInd term, $\int \rho\cos\theta d\tau = \int_0^a\int_0^\pi\int_0^{2\pi} \rho_0(a^2 - r^2)\cos^2\theta \times r^2\sin\theta dr d\theta d\phi \neq 0$.

$\Rightarrow V\alpha\frac{1}{r^2} \Rightarrow E\alpha\frac{1}{r^3}$

Q2. Two magnetic dipoles of magnitude m each are placed in a plane as shown in figure. The energy of interaction is given by



(a) Zero (b) $\frac{\mu_0 m^2}{4\pi d^3}$

(c) $\frac{3\mu_0 m^2}{2\pi d^3}$ (d) $-\frac{3\mu_0 m^2}{8\pi d^3}$

Ans: (d)

Solution: $U = \frac{\mu_0}{4\pi r^3}[\vec{m}_1\cdot\vec{m}_2 - 3(\vec{m}_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r})]$,

Since $\vec{m}_1 \perp \vec{m}_2 \Rightarrow \vec{m}_1\cdot\vec{m}_2 = 0 \Rightarrow U = \frac{\mu_0}{4\pi d^3}[-3\times m\cos 45^0 \times m\cos 45^0]$

$\Rightarrow U = -\frac{3\mu_0 m^2}{8\pi d^3}$.

Statement for Linked Answer Questions 3 and 4:

Consider the propagation of electromagnetic waves in a linear, homogeneous and isotropic material medium with electric permittivity ε and magnetic permeability μ.

Q3. For a plane wave of angular frequency ω and propagation vector $\vec{k}$ propagating in the medium Maxwell's equations reduce to

(a) $\vec{k}\cdot\vec{E}=0;\ \vec{k}\cdot\vec{H}=0;\ \vec{k}\times\vec{E}=\omega\varepsilon\vec{H};\ \vec{k}\times\vec{H}=-\omega\mu\vec{E}$

(b) $\vec{k}\cdot\vec{E}=0;\ \vec{k}\cdot\vec{H}=0;\ \vec{k}\times\vec{E}=-\omega\varepsilon\vec{H};\ \vec{k}\times\vec{H}=\omega\mu\vec{E}$

(c) $\vec{k}\cdot\vec{E}=0;\ \vec{k}\cdot\vec{H}=0;\ \vec{k}\times\vec{E}=-\omega\mu\vec{H};\ \vec{k}\times\vec{H}=\omega\varepsilon\vec{E}$

(d) $\vec{k}\cdot\vec{E}=0;\ \vec{k}\cdot\vec{H}=0;\ \vec{k}\times\vec{E}=\omega\mu\vec{H};\ \vec{k}\times\vec{H}=-\omega\varepsilon\vec{E}$

Ans: (d)

Q4. If ε and μ assume negative values in a certain frequency range, then the directions of the propagation vector $\vec{k}$ and the Poynting vector $\vec{S}$ in that frequency range are related as

(a) $\vec{k}$ and $\vec{S}$ are parallel

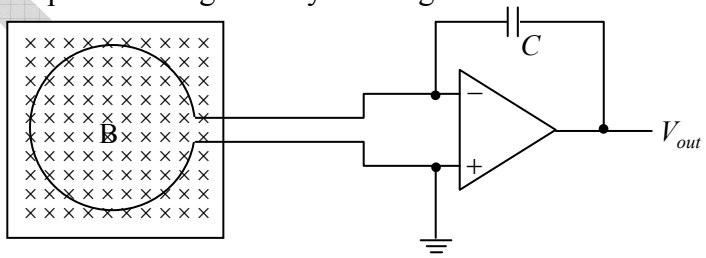
(b) $\vec{k}$ and $\vec{S}$ are anti-parallel

(c) $\vec{k}$ and $\vec{S}$ are perpendicular to each other

(d) $\vec{k}$ and $\vec{S}$ makes an angle that depends on the magnitude of $|\varepsilon|$ and $|\mu|$

Ans: (a)

Q5. Consider a conducting loop of radius a and total loop resistance R placed in a region with a magnetic field B thereby enclosing a flux ϕ_0. The loop is connected to an electronic circuit as shown, the capacitor being initially uncharged



If the loop is pulled out of the region of the magnetic field at a constant speed u, the final output voltage V_{out} is independent of

(a) ϕ_0 (b) u (c) R (d) C

Ans: (a)

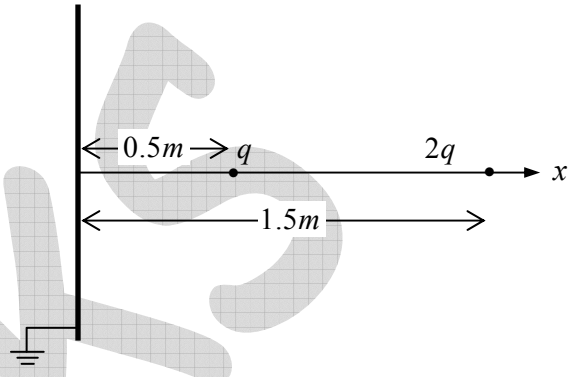
GATE-2011

Q6. If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

(a) $\vec{\nabla}\times\vec{F}=0$ (b) $\vec{\nabla}\cdot\vec{F}=0$ (c) $\vec{\nabla}V=0$ (d) $\nabla^2 V=0$

Ans: (a)

Q7. Two charges q and $2q$ are placed along the x-axis in front of a grounded, infinite conducting plane, as shown in the figure. They are located respectively at a distance of 0.5 m and 1.5 m from the plane. The force acting on the charge q is

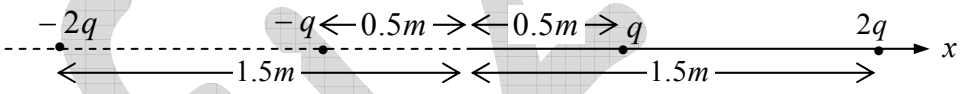


(a) $\frac{1}{4\pi\varepsilon_0}\frac{7q^2}{2}$ (b) $\frac{1}{4\pi\varepsilon_0}2q^2$

(c) $\frac{1}{4\pi\varepsilon_0}q^2$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}$

Ans: (a)

Solution: Using method of Images we can draw equivalent figure as shown below:



$$F=\frac{q}{4\pi\varepsilon_0}\left[\frac{2q}{(1)^2}+\frac{q}{(1)^2}+\frac{2q}{(2)^2}\right]=\frac{q}{4\pi\varepsilon_0}\times\frac{7q}{2}=\frac{1}{4\pi\varepsilon_0}\frac{7q^2}{2}$$

Q8. A uniform surface current is flowing in the positive y-direction over an infinite sheet lying in x-y plane. The direction of the magnetic field is

(a) along $\hat{i}$ for $z>0$ and along $-\hat{i}$ for $z<0$

(b) along $\hat{k}$ for $z>0$ and along $-\hat{k}$ for $z<0$

(c) along $-\hat{i}$ for $z>0$ and along $\hat{i}$ for $z<0$

(d) along $-\hat{k}$ for $z>0$ and along $\hat{k}$ for $z<0$

Ans: (a)

Q9. A magnetic dipole of dipole moment $\vec{m}$ is placed in a non-uniform magnetic field $\vec{B}$. If the position vector of the dipole is $\vec{r}$, the torque acting on the dipole about the origin is

(a) $\vec{r}\times\left(\vec{m}\times\vec{B}\right)$ (b) $\vec{r}\times\vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)$

(c) $\vec{m}\times\vec{B}$ (d) $\vec{m}\times\vec{B}+\vec{r}\times\nabla\left(\vec{m}\cdot\vec{B}\right)$

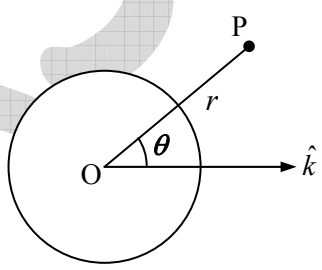
Ans: (c)

Q10. A spherical conductor of radius a is placed in a uniform electric field $\vec{E}=E_0\hat{k}$. The potential at a point $P(r, \theta)$ for $r > a$, is given by

$$\Phi(r,\theta) = \text{constant} - E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

where r is the distance of P from the centre O of the sphere and θ is the angle OP makes with the z-axis

The charge density on the sphere at $\theta = 30^o$ is



(a) $3\sqrt{3}\varepsilon_0 E_0/2$ (b) $3\varepsilon_0 E_0/2$

(c) $\sqrt{3}\varepsilon_0 E_0/2$ (d)) $\varepsilon_0 E_0/2$

Ans: (a)

Solution: $\sigma = -\varepsilon_0\left.\dfrac{\partial V}{\partial r}\right|_{r=a} = -\varepsilon_0\left[-E_0\cos\theta - \dfrac{2E_0a^3}{r^3}\cos\theta\right]_{r=a}$.

$$\sigma = -\varepsilon_0\left[-E_0\cos\theta - 2E_0\cos\theta\right] \Rightarrow \sigma = +3E_0\varepsilon_0\cos\theta = +3E_0\varepsilon_0\cos 30^0 = \frac{3\sqrt{3}}{2}\varepsilon_0 E_0$$

Q11. Which of the following expressions for a vector potential $\vec{A}$ **DOES NOT** represent a uniform magnetic field of magnitude B_0 along the z-direction?

(a) $\vec{A}=(0,B_0x,0)$ (b) $\vec{A}=(-B_0y,0,0)$

(c) $\vec{A}=\left(\dfrac{B_0x}{2},\dfrac{B_0y}{2},0\right)$ (d) $\vec{A}=\left(-\dfrac{B_0y}{2},\dfrac{B_0x}{2},0\right)$

Ans: (c)

Solution: $\vec{B}\neq\vec{\nabla}\times\vec{A}$.

Statement for Linked Questions 12 and 13:

A plane electromagnetic wave has the magnetic field given by

$$\vec{B}(x,y,z,t) = B_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{k}$$

where k is the wave number and $\hat{i}, \hat{j}$ and $\hat{k}$ are the Cartesian unit vectors in x, y and z directions respectively.

Q12. The electric field $\vec{E}(x,y,z,t)$ corresponding to the above wave is given by

(a) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$ (b) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$

(c) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{i}$ (d) $cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{j}$

Ans: (a)

Solution: $\vec{E} = -\frac{c}{k}\left(\vec{k}\times\vec{B}\right) = -\frac{c}{k}\left[-\frac{k(\hat{i}+\hat{j})}{\sqrt{2}} \times B_0 \sin\left\{\frac{(x+y)k}{\sqrt{2}} + \omega t\right\}\hat{k}\right]$

$$\vec{E} = cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$$

Q13. The average Poynting vector is given by

(a) $\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$ (b) $-\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$ (c) $\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$ (d) $-\frac{cB_0^2}{2\mu_0}\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$

Ans: (d)

Solution: $\vec{S} = \frac{cB_0^2}{2\mu_0}\hat{k} = \frac{cB_0^2}{2\mu_0} \times -\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) = \frac{-cB_0^2}{2\mu_0} \times \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$

GATE-2012

Q14. The space-time dependence of the electric field of a linearly polarized light in free space is given by $\hat{x}E_0\cos(\omega t - kz)$ where E_0, ω and k are the amplitude, the angular frequency and the wavevector, respectively. The time average energy density associated with the electric field is

(a) $\frac{1}{4}\varepsilon_0 E_0^2$ (b) $\frac{1}{2}\varepsilon_0 E_0^2$ (c) $\varepsilon_0 E_0^2$ (d) $2\varepsilon_0 E_0^2$

Ans: (a)

Solution: $u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 E^2\cos^2(wt-kz) \Rightarrow < u_E >= \frac{1}{4}\varepsilon_0 E_0^2$

Q15. A plane electromagnetic wave traveling in free space is incident normally on a glass plate of refractive index 3/2. If there is no absorption by the glass, its reflectivity is

(a) 4% (b) 16% (c) 20% (d) 50%

Ans: (a)

Solution: $R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1-3/2}{1+3/2}\right)^2 = \frac{1}{4}\times\frac{4}{25} = .04 \; or \; 4\%$

Q16. The electric and the magnetic field $\vec{E}(z,t)$ and $\vec{B}(z,t)$, respectively corresponding to the scalar potential $\phi(z,t) = 0$ and vector potential $\vec{A}(z,t) = \hat{i}tz$ are

(a) $\vec{E} = \hat{i}z$ and $\vec{B} = -\hat{j}t$ (b) $\vec{E} = \hat{i}z$ and $\vec{B} = \hat{j}t$

(c) $\vec{E} = -\hat{i}z$ and $\vec{B} = -\hat{j}t$ (d) $\vec{E} = -\hat{i}z$ and $\vec{B} = -\hat{j}t$

Ans: (d)

Solution: $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} = -\frac{\partial\vec{A}}{\partial t} = -\hat{i}z, \;\; \vec{B} = \vec{\nabla}\times\vec{A} = +\hat{j}t.$

Q17. A plane polarized electromagnetic wave in free space at time t=0 is given by $\vec{E}(x,z) = 10\hat{j}\exp\left[i(6x+8z)\right]$. The magnetic field $\vec{B}(x,z,t)$ is given by

(a) $\vec{B}(x,z,t) = \frac{1}{c}\left(6\hat{k} - 8\hat{i}\right)\exp[i(6x+8z-10ct)]$

(b) $\vec{B}(x,z,t) = \frac{1}{c}\left(6\hat{k} + 8\hat{i}\right)\exp[i(6x+8z-10ct)]$

(c) $\vec{B}(x,z,t) = \frac{1}{c}\left(6\hat{k} - 8\hat{i}\right)\exp[i(6x+8z-ct)]$

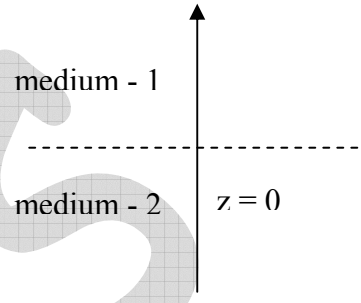
(d) $\vec{B}(x,z,t) = \frac{1}{c}\left(6\hat{k} + 8\hat{i}\right)\exp[i(6x+8z+ct)]$

Ans: (a)

Solution: $\vec{B} = \frac{1}{c}\left(\hat{k} \times \vec{E}\right) = \frac{1}{c}\left(\frac{\vec{k}}{|\vec{k}|} \times \vec{E}\right) = \frac{1}{c}\left(\frac{6\hat{i} + 8\hat{k}}{10}\right) \times 10\,\hat{j} \exp\left[i\left(\vec{k}.\vec{r} - \omega t\right)\right]$

$\vec{B} = \frac{1}{c}\left(6\hat{k} - 8\hat{i}\right)\exp\left[i(6x + 8z - 10ct)\right], \quad \omega = 10c.$

Q18. Two infinitely extended homogeneous isotopic dielectric media (medium-1and medium-2 with dielectric constant $\frac{\varepsilon_1}{\varepsilon_0} = 2$ and $\frac{\varepsilon_2}{\varepsilon_0} = 5$, respectively) meet at the z = 0 plane as shown in the figure. A uniform electric field exists everywhere. For $z \geq 0$, the electric field is given by $\vec{E}_1 = 2\hat{i} - 3\hat{j} + 5\hat{k}$. The interface separating the two media is charge free. The electric displacement vector in the medium-2 is given by



(a) $\vec{D}_2 = \varepsilon_0\left[10\hat{i} + 15\hat{j} + 10\hat{k}\right]$ (b) $\vec{D}_2 = \varepsilon_0\left[10\hat{i} - 15\hat{j} + 10\hat{k}\right]$

(c) $\vec{D}_2 = \varepsilon_0\left[4\hat{i} - 6\hat{j} + 10\hat{k}\right]$ (d) $\vec{D}_2 = \varepsilon_0\left[4\hat{i} + 6\hat{j} + 10\hat{k}\right]$

Ans: (b)

Solution: $\because E_1^{II} = E_2^{II} \Rightarrow E_2^{II} = 2\hat{i} - 3\hat{j}$

and $\sigma_f = 0 \Rightarrow D_1^{\perp} = D_2^{\perp} \Rightarrow E_2^{\perp} = \frac{\varepsilon_1}{\varepsilon_2} E_1^{\perp} = \frac{2 \times 5}{5}\hat{k} = 2\hat{k} \Rightarrow \vec{E}_2 = 2\hat{i} - 3\hat{j} + 2\hat{k}$

$\Rightarrow \vec{D}_2 = \varepsilon_2 \vec{E}_2 = \varepsilon_0\left[10\hat{i} - 15\hat{j} + 10\hat{k}\right].$

GATE-2013

Q19. At a surface current, which one of the magnetostatic boundary condition is **NOT** CORRECT?

(a) Normal component of the magnetic field is continuous.

(b) Normal component of the magnetic vector potential is continuous.

(c) Tangential component of the magnetic vector potential is continuous.

(d) Tangential component of the magnetic vector potential is not continuous.

Ans: (d)

Q20. Interference fringes are seen at an observation plane $z=0$, by the superposition of two plane waves $A_1 \exp\left[i\left(\vec{k}_1 \cdot \vec{r} - \omega t\right)\right]$ and $A_2 \exp\left[i\left(\vec{k}_2 \cdot \vec{r} - \omega t\right)\right]$, where A_1 and A_2 are real amplitudes. The condition for interference maximum is

(a) $\left(\vec{k}_1 - \vec{k}_2\right)\cdot \vec{r} = (2m+1)\pi$ (b) $\left(\vec{k}_1 - \vec{k}_2\right)\cdot \vec{r} = 2m\pi$

(c) $\left(\vec{k}_1 + \vec{k}_2\right)\cdot \vec{r} = (2m+1)\pi$ (d) $\left(\vec{k}_1 + \vec{k}_2\right)\cdot \vec{r} = 2m\pi$

Ans: (b)

Q21. For a scalar function φ satisfying the Laplace equation, $\vec{\nabla}\varphi$ has

(a) zero curl and non-zero divergence

(b) non-zero curl and zero divergence

(c) zero curl and zero divergence

(d) non-zero curl and non-zero divergence

Ans: (c)

Solution: $\nabla^2\varphi = 0 \Rightarrow \vec{\nabla}.\left(\vec{\nabla}\varphi\right) = 0$ and $\Rightarrow \vec{\nabla}\times\left(\vec{\nabla}\varphi\right) = 0$.

Q22. A circularly polarized monochromatic plane wave is incident on a dielectric interface at Brewaster angle. Which one of the following statements is correct?

(a) The reflected light is plane polarized in the plane of incidence and the transmitted light is circularly polarized.

(b) The reflected light is plane polarized perpendicular to the plane of incidence and the transmitted light is plane polarized in the plane of incidence.

(c) The reflected light is plane polarized perpendicular to the plane of incidence and the transmitted light is elliptically polarized.

(d) There will be no reflected light and the transmitted light is circularly polarized.

Ans: (c)

Q23. A charge distribution has the charge density given by $\rho = Q\{\delta(x-x_0) - \delta(x+x_0)\}$. For this charge distribution the electric field at $(2x_0, 0, 0)$

(a) $\frac{2Q\hat{x}}{9\pi\varepsilon_0 x_0^2}$ (b) $\frac{Q\hat{x}}{4\pi\varepsilon_0 x_0^3}$ (c) $\frac{Q\hat{x}}{4\pi\varepsilon_0 x_0^2}$ (d) $\frac{Q\hat{x}}{16\pi\varepsilon_0 x_0^2}$

Ans:

Solution: Potential $V(r)=\frac{1}{4\pi\varepsilon_0}\left[\int_{-a}^{a}\frac{\rho(x')}{x}dx'+\int_{-a}^{a}\frac{\rho(x')}{x^2}x'dx'+\int_{-a}^{a}\frac{\rho(x')}{x^3}x'^2dx'+....\right]$

First term, total charge

$$Q_T=\int\rho(x')dx'=Q\int_{-x_0}^{x_0}\delta(x'-x_0)dx'-Q\int_{-x_0}^{x_0}\delta(x'+x_0)dx'=Q-Q=0$$

Second term, dipole moment

$$p=\int x'\rho(x')dx'=Q\int_{-x_0}^{x_0}x'\delta(x'-x_0)dx'-Q\int_{-x_0}^{x_0}x'\delta(x'+x_0)dx'=Qx_0-Q\times-x_0=2Qx_0$$

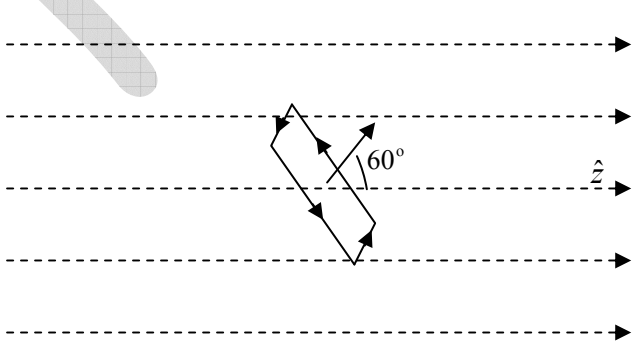
$$V=\frac{2Qx_0}{4\pi\varepsilon_0x^2}\Rightarrow\vec{E}=-\frac{\partial V}{\partial x}\hat{x}=\frac{4Qx_0}{4\pi\varepsilon_0x^3}\hat{x}=\frac{4Qx_0}{4\pi\varepsilon_0(2x_0)^3}\hat{x}=\frac{Q}{8\pi\varepsilon_0x_0^2}\hat{x}$$

Q24. A monochromatic plane wave at oblique incidence undergoes reflection at a dielectric interface. If $\hat{k}_i,\hat{k}_r$ and $\hat{n}$ are the unit vectors in the directions of incident wave, reflected wave and the normal to the surface respectively, which one of the following expressions is correct?

(a) $(\hat{k}_i-\hat{k}_r)\times\hat{n}\neq0$ (b) $(\hat{k}_i-\hat{k}_r)\cdot\hat{n}=0$ (c) $(\hat{k}_i\times\hat{n})\cdot\hat{k}_r=0$ (d) $(\hat{k}_i\times\hat{n})\cdot\hat{k}_r\neq0$

Ans: (c)

Q25. In a constant magnetic field of 0.6 Tesla along the z direction, find the value of the path integral $\oint\vec{A}\cdot\vec{dl}$ in the units of (Tesla m^2) on a square loop of side length $(1/\sqrt{2})$ meters. The normal to the loop makes an angle of 60^0 to the z-axis, as shown in the figure. The answer should be up to two decimal places. _____



Ans: 0.15

Solution: $\oint \vec{A}\cdot\vec{dl} = \int_S \left(\vec{\nabla}\times\vec{A}\right)d\vec{a} = \int_S \vec{B}.d\vec{a} = BA\cos 60^0 = 0.6\times\left(\frac{1}{\sqrt{2}}\right)^2\times\frac{1}{2} = 0.15T.m^2$

GATE-2014

Q26. Which one of the following quantities is invariant under Lorentz transformation?

(a) Charge density (b) Charge (c) Current (d) Electric field

Ans: (b)

Q27. An unpolarized light wave is incident from air on a glass surface at the Brewster angle. The angle between the reflected and the refracted wave is

(a) 0^o (b) 45^o (c) 90^o (d) 120^o

Ans: (c)

Q28. The electric field of a uniform plane wave propagating in a dielectric non-conducting medium is given by $\vec{E} = \hat{x}10\cos\left(6\pi\times10^7 t - 0.4\pi z\right)\ V/m$. The phase velocity of the wave is _____ $10^8\, m/s$

Ans: 1.5

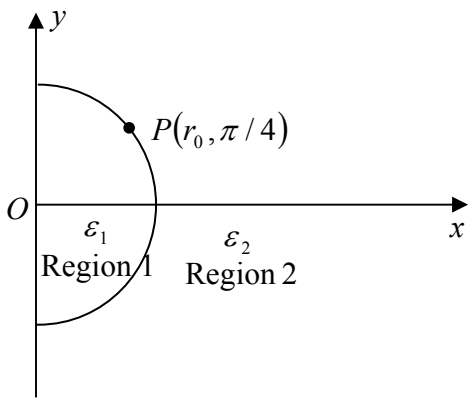
Solution: $v = \frac{\omega}{k} = \frac{6\pi\times10^7}{0.4\pi} = 1.5\times10^8\ m/\sec$

Q29. If the vector potential $\vec{A} = \alpha x\hat{x} + 2y\hat{y} - 3z\hat{z}$, satisfies the Coulomb gauge, the value of the constant α is _____

Ans: 1

Solution: Coulomb gauge condition $\vec{\nabla}.\vec{A} = 0 \Rightarrow \alpha + 2 - 3 = 0 \Rightarrow \alpha = 1$

Q30. A ray of light inside Region 1 in the xy-plane is incident at the semicircular boundary that carries no free charges. The electric field at the point $P\left(r_0, \frac{\pi}{4}\right)$ in plane polar coordinates is $\vec{E}_1 = 7\hat{e}_r - 3\hat{e}_\varphi$ where $\hat{e}_r$ and $\hat{e}_\varphi$ are the unit vectors. The emerging ray in Region 2 has the electric field $\vec{E}_2$ parallel to x-axis. If ε_1 and ε_2 are the dielectric constants of Region-1 and Region-2 respectively, then $\frac{\varepsilon_2}{\varepsilon_1}$ is _____

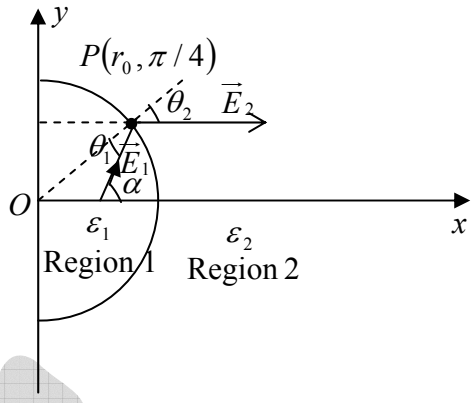


Ans: 2.32

Solution: $\because \vec{E}_1 = 7\hat{e}_r - 3\hat{e}_\varphi$

$$\Rightarrow E_x = \left(7\hat{e}_r - 3\hat{e}_\varphi\right).\hat{x} = 7\cos 45 + 3\sin 45 = \frac{10}{\sqrt{2}}$$

$$\Rightarrow E_y = \left(7\hat{e}_r - 3\hat{e}_\varphi\right).\hat{y} = 7\sin 45 - 3\sin 45 = \frac{4}{\sqrt{2}}$$



Thus $\vec{E}_1$ makes an angle $\alpha = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left(\frac{4}{10}\right) = 21.8^0$

$\because \frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_2}{\varepsilon_1} \Rightarrow \frac{\varepsilon_2}{\varepsilon_1} = \frac{\tan 45}{\tan 23.2} = 2.32$. where $\theta_1 = \alpha - 45^0$ and $\theta_2 = 45^0$

Q31. The value of the magnetic field required to maintain non-relativistic protons of energy $1MeV$ in a circular orbit of radius 100 mm is _____Tesla

(Given: $m_p = 1.67\times10^{-27}\,kg$, $e = 1.6\times10^{-19}\,C$)

Ans: 1.44

Solution: $E = \frac{q^2B^2R^2}{2m_p} \Rightarrow 1.6\times10^{-13} = \frac{\left(1.6\times10^{-19}\right)^2 B^2 \left(0.1\right)^2}{2\left(1.67\times10^{-27}\right)} \Rightarrow B^2 = \frac{1.6\times10^{-13}\times2\left(1.67\times10^{-27}\right)}{\left(1.6\times10^{-19}\right)^2\left(0.1\right)^2}$

$$\Rightarrow B^2 = \frac{10^{-13}\times2\left(1.67\times10^{-27}\right)}{\left(1.6\times10^{-38}\right)\left(0.01\right)} = \frac{3.34\times10^{-40}}{1.6\times10^{-40}} = 2.08 \Rightarrow B = \sqrt{2.08}\,Tesla = 1.44Tesla$$

Q32. In an interference pattern formed by two coherent sources, the maximum and minimum intensities are $9I_0$ and I_0 respectively. The intensities of the individual wave are

(a) $3I_0$ and I_0 (b) $4I_0$ and I_0 (c) $5I_0$ and $4I_0$ (d) $9I_0$ and I_0

Ans: (b)

Solution: $I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$ and $I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$

$9I_0 = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$ and $I_0 = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 \Rightarrow I_1 = 4I_0$ *and* $I_2 = I_0$

Q33. The intensity of a laser in free space is $150mW/m^2$. The corresponding amplitude of the electric field of the laser is _____ $\frac{V}{m}$ $\left(\varepsilon_0 = 8.854\times10^{-12}C^2/N.m^2\right)$

Ans: 10.6

Solution: $I = \frac{1}{2} c\varepsilon_0 E_0^2 \Rightarrow E_0 = \sqrt{\frac{2I}{c\varepsilon_0}} = \sqrt{\frac{2\times150\times10^{-3}}{3\times10^8\times8.854\times10^{-12}}} = 10.6\,V/m$

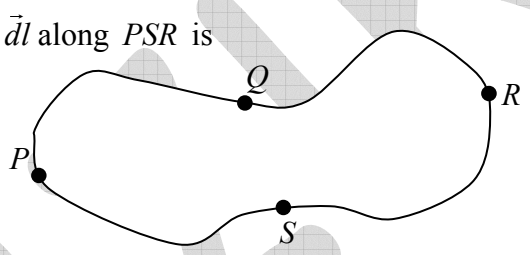
GATE-2015

Q34. A point charge is placed between two semi-infinite conducting plates which are inclined at an angle of 30^o with respect to each other. The number of image charges is_____.

Ans.: 11

Solution: $n = \frac{360}{\theta} - 1 = \frac{360}{30} - 1 = 11$

Q35. Given that the magnetic flux through the closed loop $PQRSP$ is ϕ. If $\int_P^R \vec{A}\cdot\vec{dl} = \phi_1$ along PQR, the value of $\int_P^R \vec{A}\cdot\vec{dl}$ along PSR is



(a) $\phi - \phi_1$ (b) $\phi_1 - \phi$ (c) $-\phi_1$ (d) ϕ_1

Ans.: (b)

Solution: $\phi = \int_s \vec{B}.d\vec{a} = \oint \vec{A}.d\vec{l} = \int_P^R \vec{A}\cdot d\vec{l} + \int_R^P \vec{A}\cdot d\vec{l} \Rightarrow \phi = \phi_1 - \int_P^R \vec{A}\cdot d\vec{l} \Rightarrow \int_P^R \vec{A}\cdot d\vec{l} = \phi_1 - \phi$

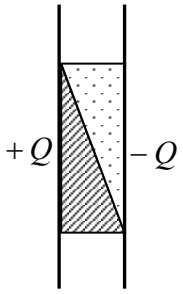
Q36. The space between two plates of a capacitor carrying charges $+Q$ and $-Q$ is filled with two different dielectric materials, as shown in the figure. Across the interface of the two dielectric materials, which one of the following statements is correct?

(a) $\vec{E}$ and $\vec{D}$ are continuous

(b) $\vec{E}$ is continuous and $\vec{D}$ is discontinuous

(c) $\vec{D}$ is continuous and $\vec{E}$ is discontinuous

(d) $\vec{E}$ and $\vec{D}$ are discontinuous



Ans.: (d)

Q37. Four forces are given below in Cartesian and spherical polar coordinates

(i) $\vec{F}_1 = K\exp\left(\frac{-r^2}{R^2}\right)\hat{r}$ (ii) $\vec{F}_2 = K\left(x^3\hat{y} - y^3\hat{z}\right)$

(iii) $\vec{F}_3 = K\left(x^3\hat{x} + y^3\hat{y}\right)$ (iv) $\vec{F}_4 = K\left(\frac{\hat{\phi}}{r}\right)$

where K is a constant Identify the correct option

(a) (iii) and (iv) are conservative but (i) and (ii)are not

(b) (i) and (ii) are conservative but (iii) and (iv) are not

(c) (ii) and (iii) are conservative but (i) and (iv) are not

(d) (i) and (iii) are conservative but (ii) and (iv) are not

Ans.: (d)

Solution: $\vec{\nabla}\times\vec{F}_1 = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ k\exp\left(-\frac{r^2}{R^2}\right) & 0 & 0 \end{vmatrix} = 0$

$$\vec{\nabla}\times\vec{F}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & kx^3 & -ky^3 \end{vmatrix} = \hat{x}\left(-3ky^2 - 0\right) + \hat{z}\left(3kx^2 - 0\right) = -3ky^2\hat{x} + 3kx^2\hat{z}$$

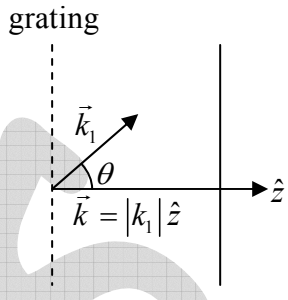
$$\vec{\nabla}\times\vec{F}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx^3 & ky^3 & 0 \end{vmatrix} = 0$$

$$\vec{\nabla}\times\vec{F}_4 = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\times\frac{k}{r} \end{vmatrix} = \hat{r}\left[k\cos\theta\right]\times\frac{1}{r^2\sin\theta}$$

Q38. A monochromatic plane wave (wavelength $= 600\,nm$) $E_0 \exp\left[i(kz - \omega t)\right]$ is incident normally on a diffraction grating giving rise to a plane wave $E_1 \exp\left[i(\vec{k}_1 \cdot \vec{r} - \omega t)\right]$ in the first order of diffraction. Here $E_1 < E_0$ *and* $\vec{k}_1 = |\vec{k}_1|\left[\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z}\right]$. The period (in μm) of the diffraction grating is _____ (upto one decimal place)

Ans.: 1.2

Solution: $d\sin\theta = n\lambda \Rightarrow d = \dfrac{\lambda}{\sin\theta} \qquad \because n = 1$



and $\vec{k}_1 = |\vec{k}_1|\left[\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z}\right]$

$$\Rightarrow \sin\theta = \frac{\vec{k} \times \vec{k}_1}{|\vec{k}_1||\vec{k}|} = \frac{\hat{z} \times \left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z}\right)}{\sqrt{\frac{1}{4} + \frac{3}{4}} \times \sqrt{\frac{1}{4} + \frac{3}{4}}} = \frac{1}{2} \Rightarrow \theta = 30^0$$

$$\Rightarrow d = \frac{600}{\sin 30} nm = 1200\ nm = 1.2\ \mu m$$

Q39. A long solenoid is embedded in a conducting medium and is insulated from the medium. If the current through the solenoid is increased at a constant rate, the induced current in the medium as a function of the radial distance r from the axis of the solenoid is proportional to

(a) r^2 inside the solenoid and $\frac{1}{r}$ outside (b) r inside the solenoid and $\frac{1}{r^2}$ outside

(c) r^2 inside the solenoid and $\frac{1}{r^2}$ outside (d) r inside the solenoid and $\frac{1}{r}$ outside

Ans.: (d)

Solution: $\oint \vec{E} \cdot d\vec{l} = -\int \dfrac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$;

$$For \quad r < R, \ |\vec{E}|2\pi r = -\mu_0 n \frac{dI}{dt} \int_{r'=0}^{r} 2\pi r' dr' = -\mu_0 n \frac{dI}{dt} \frac{2\pi r^2}{2} \Rightarrow |\vec{E}| = -\frac{1}{2}\mu_0 n \frac{dI}{dt} r$$

$$For \quad r > R, \ |\vec{E}|2\pi r = -\mu_0 n \frac{dI}{dt} \int_{r'=0}^{R} 2\pi r' dr' = -\mu_0 n \frac{dI}{dt} \frac{2\pi R^2}{2} \Rightarrow |\vec{E}| = -\frac{1}{2r}\mu_0 n \frac{dI}{dt} R^2$$

Q40. A plane wave $(\hat{x}+i\hat{y})E_0\exp[i(kz-\omega t)]$ after passing through an optical element emerges as $(\hat{x}-i\hat{y})E_0\exp[i(kz-\omega t)]$, where k and ω are the wavevector and the angular frequency, respectively. The optical element is a

(a) quarter wave plate (b) half wave plate

(c) polarizer (d) Faraday rotator

Ans.: (b)

Solution: Incident wave: $\left(\hat{x}+i\hat{y}\right)E_0e^{i\theta}=\left[E_0\cos\theta\hat{x}-E_0\sin\theta\hat{y}\right]$

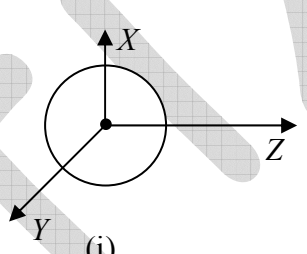
Left circular polarization with phase angle $\phi_1=-\theta=\theta e^{i\pi}$

Emergent wave: $\left(\hat{x}-i\hat{y}\right)E_0e^{i\theta}=\left[E_0\cos\theta\hat{x}+E_0\sin\theta\hat{y}\right]$

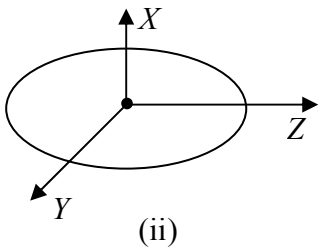
Right circular polarization with phase angle $\phi_1=+\theta=\theta e^{i0}$

Thus there is phase change of π and hence path difference is $\frac{\lambda}{2}$.

Q41. A charge $-q$ is distributed uniformly over a sphere, with a positive charge q at its center in (i). Also in (ii), a charge $-q$ is distributed uniformly over an ellipsoid with a positive charge q at its center. With respect to the origin of the coordinate system, which one of the following statements is correct?



(i)



(ii)

(a) The dipole moment is zero in both (i) and (ii)

(b) The dipole moment is non-zero in (i) but zero in (ii)

(c) The dipole moment is zero in (i) but non-zero in (ii)

(d) The dipole moment is non-zero in both (i) and (ii)

Ans.: (a)

Solution: $\vec{p}=\sum q_i\vec{r}_i=0$ in both cases.

GATE-2016

Q42. Which of the following magnetic vector potentials gives rise to a uniform magnetic field $B_0\hat{k}$?

(a) $B_0 z\,\hat{k}$ (b) $-B_0 x\,\hat{j}$ (c) $\frac{B_0}{2}\left(-y\hat{i}+x\hat{j}\right)$ (d) $\frac{B_0}{2}\left(y\hat{i}+x\hat{j}\right)$

Ans.: (c)

Solution: (a) $\vec{\nabla}\times\vec{A}=0$

(b) $\vec{\nabla}\times\vec{A}=-B_0\hat{k}$

(c) $\vec{\nabla}\times\vec{A}=B_0\hat{k}$

(d) $\vec{\nabla}\times\vec{A}=0$

Q43. The magnitude of the magnetic dipole moment associated with a square shaped loop carrying a steady current I is m. If this loop is changed to a circular shape with the same current I passing through it, the magnetic dipole moment becomes $\frac{pm}{\pi}$. The value of p is _____.

Ans.: 4

Solution: Magnetic dipole moment associated with a square shaped loop (let side is a) carrying a steady current I is $m=Ia^2$.

Magnetic dipole moment associated with a circular shaped loop (let radius is r) carrying a steady current I is $m'=I\pi r^2$.

Here $4a=2\pi r\Rightarrow r=\frac{2a}{\pi}\Rightarrow m'=I\pi r^2=I\pi\left(\frac{2a}{\pi}\right)^2=\frac{4Ia^2}{\pi}=\frac{4m}{\pi}$

Q44. In a Young's double slit experiment using light, the apparatus has two slits of unequal widths. When only slit-1 is open, the maximum observed intensity on the screen is $4I_0$. When only slit-2 is open, the maximum observed intensity is I_0. When both the slits are open, an interference pattern appears on the screen. The ratio of the intensity of the principal maximum to that of the nearest minimum is _____.

Ans.: 9

Solution: $\frac{I_{\max}}{I_{\min}}=\frac{\left(\sqrt{I_1}+\sqrt{I_2}\right)^2}{\left(\sqrt{I_1}-\sqrt{I_2}\right)^2}=\frac{\left(\sqrt{4I_0}+\sqrt{I_0}\right)^2}{\left(\sqrt{4I_0}-\sqrt{I_0}\right)^2}=\frac{\left(2\sqrt{I_0}+\sqrt{I_0}\right)^2}{\left(2\sqrt{I_0}-\sqrt{I_0}\right)^2}=\frac{9I_0}{I_0}=9$

Q45. An infinite, conducting slab kept in a horizontal plane carries a uniform charge density σ. Another infinite slab of thickness *t*, made of a linear dielectric material of dielectric constant k, *is* kept above the conducting slab. The bound charge density on the upper surface of the dielectric slab is

(a) $\frac{\sigma}{2k}$ (b) $\frac{\sigma}{k}$ (c) $\frac{\sigma(k-2)}{2k}$ (d) $\frac{\sigma(k-1)}{k}$

Ans.: (d)

Solution:

Electric field due to infinite, conducting slab inside the dielectric is $\vec{E}=\frac{\sigma}{\varepsilon}\hat{z}=\frac{\sigma}{\varepsilon_0 k}\hat{z}$

Polarisation $\vec{P}=\varepsilon_0\chi_e\vec{E}=\varepsilon_0(k-1)\frac{\sigma}{\varepsilon_0 k}\hat{z}=\frac{\sigma(k-1)}{k}\hat{z}\Rightarrow\sigma_1=\vec{P}.\hat{z}=\frac{\sigma(k-1)}{k}$

Q46. The electric field component of a plane electromagnetic wave travelling in vacuum is given by $\vec{E}(z,t)=E_0\cos(kz-\omega t)\hat{i}$. The Poynting vector for the wave is

(a) $\left(\frac{c\varepsilon_0}{2}\right)E_0^2\cos^2(kz-\omega t)\hat{j}$ (b) $\left(\frac{c\varepsilon_0}{2}\right)E_0^2\cos^2(kz-\omega t)\hat{k}$

(c) $c\varepsilon_0E_0^2\cos^2(kz-\omega t)\hat{j}$ (d) $c\varepsilon_0E_0^2\cos^2(kz-\omega t)\hat{k}$

Ans.: (d)

Solution: $\vec{E}(z,t)=E_0\cos(kz-\omega t)\hat{i}\Rightarrow\vec{B}=\frac{1}{c}\hat{z}\times\vec{E}(z,t)=\frac{E_0}{c}\cos(kz-\omega t)\hat{j}$

The Poynting vector for the wave is

$$\vec{S}=\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)=\frac{E_0^2}{\mu_0 c}\cos^2(kz-\omega t)\hat{k}=c\varepsilon_0E_0^2\cos^2(kz-\omega t)\hat{k}$$

Q47. The $x-y$ plane is the boundary between free space and a magnetic material with relative permeability μ_r. The magnetic field in the free space is $B_x\hat{i}+B_z\hat{k}$. The magnetic field in the magnetic material is

(a) $B_x\hat{i}+B_z\hat{k}$ (b) $B_x\hat{i}+\mu_rB_z\hat{k}$ (c) $\frac{1}{\mu_r}B_x\hat{i}+B_z\hat{k}$ (d) $\mu_rB_x\hat{i}+B_z\hat{k}$

Ans.: (d)

Solution: $B_1^{\perp} = B_z\hat{k} = B_2^{\perp}$ and $H_1^{\parallel} = H_2^{\parallel} \Rightarrow \frac{B_1^{\parallel}}{\mu_0} = \frac{B_2^{\parallel}}{\mu_0\mu_r} \Rightarrow B_2^{\parallel} = \mu_r B_1^{\parallel} = \mu_r B_x\hat{i}$

The magnetic field in the magnetic material is $\mu_r B_x\hat{i} + B_z\hat{k}$

GATE- 2017

Q48. Identical charges q are placed at five vertices of a regular hexagon of side a. The magnitude of the electric field and the electrostatic potential at the centre of the hexagon are respectively

(a) $0, 0$

(b) $\frac{q}{4\pi\varepsilon_0 a^2}, \frac{q}{4\pi\varepsilon_0 a}$

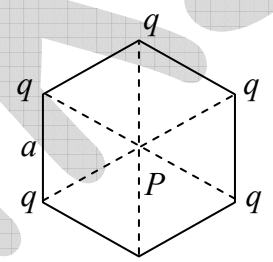
(c) $\frac{q}{4\pi\varepsilon_0 a^2}, \frac{5q}{4\pi\varepsilon_0 a}$

(d) $\frac{\sqrt{5}q}{4\pi\varepsilon_0 a^2}, \frac{\sqrt{5}q}{4\pi\varepsilon_0 a}$

Ans. : (c)

Solution: The resultant field at P is $E = \frac{q}{4\pi\varepsilon_0 a^2}$

The electrostatic potential at P is $V = \frac{5q}{4\pi\varepsilon_0 a}$



Q49. A parallel plate capacitor with square plates of side $1m$ separated by 1 micro meter is filled with a medium of dielectric constant of 10. If the charges on the two plates are $1C$ and $-1C$, the voltage across the capacitor is…….. kV. (up to two decimal places). ($\varepsilon_0 = 8.854\times10^{-12}\ F/m$)

Ans. : 11.29

Solution: $q = CV = \frac{\varepsilon_0\varepsilon_r A}{d}V \Rightarrow V = \frac{qd}{\varepsilon_0\varepsilon_r A} = \frac{1\times1\times10^{-6}}{8.854\times10^{-12}\times10\times1} \approx 11.29kV$

Q50. Light is incident from a medium of refractive index $n = 1.5$ onto vacuum. The smallest angle of incidence for which the light is not transmitted into vacuum is…….. degrees. (up to two decimal places)

Ans. : 41.8

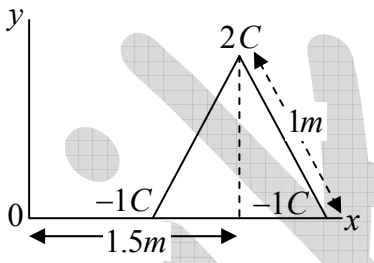
Solution: $\sin\theta_C = \frac{n_2}{n_1} = \frac{1}{1.5} \Rightarrow \theta_C = \sin^{-1}\left(\frac{1}{1.5}\right) \Rightarrow \theta_C = 41.8$

Q51. A monochromatic plane wave in free space with electric field amplitude of $1\,V/m$ is normally incident on a fully reflecting mirror. The pressure exerted on the mirror is……$\times 10^{-12}\,Pa$. (up to two decimal places) ($\varepsilon_0 = 8.854\times10^{-12}\,F/m$)

Ans. : 8.85

Solution: $P = \dfrac{2I}{c} = \dfrac{2}{c}\times\dfrac{1}{2}c\varepsilon_0 E_0^2 = \varepsilon_0 E_0^2 = 8.854\times10^{-12}\times(1)^2 = 8.85\times10^{-12}\,Pa$

Q52. Three charges $(2C,-1C,-1C)$ are placed at the vertices of an equilateral triangle of side $1m$ as shown in the figure. The component of the electric dipole moment about the marked origin along the $\hat{y}$ direction is……$C\,m$.



Ans. : 1.73

Solution: $\vec{p} = -1(1\hat{x}) - 1(2\hat{x}) + 2\left(1.5\hat{x} + \sqrt{1-0.25}\,\hat{y}\right)$

Along the $\hat{y}$ direction $= 2\times\sqrt{1-0.25} = 1.73$

Q53. An infinite solenoid carries a time varying current $I(t) = At^2$, with $A \neq 0$. The axis of the solenoid is along the $\hat{z}$ direction. $\hat{r}$ and $\hat{\theta}$ are the usual radial and polar directions in cylindrical polar coordinates. $\vec{B} = B_r\hat{r} + B_\theta\hat{\theta} + B_z\hat{z}$ is the magnetic field at a point outside the solenoid. Which one of the following statements is true?

(a) $B_r = 0, B_\theta = 0, B_z = 0$ (b) $B_r \neq 0, B_\theta \neq 0, B_z = 0$

(c) $B_r \neq 0, B_\theta \neq 0, B_z \neq 0$ (d) $B_r = 0, B_\theta = 0, B_z \neq 0$

Ans. : (d)

Q54. A uniform volume charge density is placed inside a conductor (with resistivity $10^{-2}\,\Omega m$). The charge density becomes $\dfrac{1}{(2.718)}$ of its original value after time…….Fermi seconds (up to two decimal places) ($\varepsilon_0 = 8.854\times10^{-12}\,F/m$)

Ans. : 88.54

Solution: $\rho(t)=\rho(0)e^{-\sigma t/\varepsilon_0} \Rightarrow -\sigma t/\varepsilon_0 = \ln\frac{\rho(t)}{\rho(0)} = \ln\frac{1}{2.718} = 1$

$$\Rightarrow t = \frac{\varepsilon_0}{\sigma} = 8.854\times10^{-12}\times10^{-2} = 88.54\times10^{-15}\ \text{sec} = 88.54\,fs$$

Q55. Consider a metal with free electron density of $6\times10^{22}\,cm^{-3}$. The lowest frequency of electromagnetic radiation to which this metal is transparent, is $1.38\times10^{16}\,Hz$. If this metal had a free electron density of $1.8\times10^{23}\,cm^{-3}$ instead, the lowest frequency electromagnetic radiation to which it would be transparent is……$\times10^{16}\,Hz$ (up to two decimal places).

Ans. : 2.39

Solution: Cut-off frequency is $f \propto \sqrt{n}$.

Thus $\frac{f_2}{f_1} = \sqrt{\frac{n_2}{n_1}} \Rightarrow f_2 = f_1\sqrt{\frac{n_2}{n_1}} \Rightarrow f_2 = 1.38\times10^{16}\sqrt{\frac{1.8\times10^{23}}{6\times10^{22}}} = 2.39\times10^{16}\,Hz$

GATE- 2018

Q56. Among electric field ($\vec{E}$), magnetic field ($\vec{B}$), angular momentum ($\vec{L}$) and vector potential ($\vec{A}$), which is/are **odd** under parity (space inversion) operation?

(a) $\vec{E}$ only (b) $\vec{E}$ and $\vec{A}$ only

(c) $\vec{E}$ and $\vec{B}$ only (d) $\vec{B}$ and $\vec{L}$ only

Ans. : (b)

Solution: Under parity operation $r \to -r$

$E = -\frac{\partial V}{\partial r}$; $E : P \to -E$

$B = \vec{I}\times\vec{r}$; $B : P \to +B$

$L = \vec{r}\times\vec{p}$; $L : P \to +L$

$E = -\frac{\partial A}{\partial t}$; $A : P \to -A$

Q57. An infinitely long straight wire is carrying a steady current I. The ratio of magnetic energy density at distance r_1 to that at $r_2 (= 2r_1)$ from the wire is _____.

Ans. : 4

Solution: $u_B = \dfrac{B^2}{2\mu_0} \propto \dfrac{1}{r^2} \Rightarrow \dfrac{u_{B1}}{u_{B2}} = \dfrac{r_2^2}{r_1^2} = \dfrac{(2r_1)}{r_1^2} = 4$

Q58. A light beam of intensity I_0 is falling normally on a surface. The surface absorbs 20% of the intensity and the rest is reflected. The radiation pressure on the surface is given by $X I_0 / c$, where X is _____ (up to one decimal place). Here c is the speed of light.

Ans. : 1.8

Solution: Radiation pressure $= \dfrac{I_0}{c} - \left(-0.8\dfrac{I_0}{c}\right) = 1.8\dfrac{I_0}{c}$

Q59. The number of independent components of a general electromagnetic field tensor is_____

Ans. : 6

Solution: In Cartesian co-ordinate, three Independent coordinate for electric field, (E_x, E_y, E_z) and three Independent co-ordinate for magnetic field (B_x, B_y, B_z).

Q60. Consider an infinitely long solenoid with N turns per unit length, radius R and carrying a current $I(t) = \alpha \cos \omega t$, where α is a constant and ω is the angular frequency. The magnitude of electric field at the surface of the solenoid is

(a) $\dfrac{1}{2}\mu_0 NR\omega\alpha \sin \omega t$ (b) $\dfrac{1}{2}\mu_0 \omega NR \cos \omega t$

(c) $\mu_0 NR\omega\alpha \sin \omega t$ (d) $\mu_0 \omega NR \cos \omega t$

Ans. : (a)

Solution: $\vec{B} = \begin{cases} \mu_0 NI(t)\hat{z}, & \text{inside} \\ 0 & \text{, outside} \end{cases}$

Since, $\oint\limits_{line} \vec{E} \cdot d\vec{l} = -\int \dfrac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$

$\Rightarrow |\vec{E}| \times 2\pi R = -\mu_0 N(-\alpha\omega \sin \omega t) \times \pi R^2$

$$\Rightarrow \left|\vec{E}\right| = \frac{1}{2}\mu_0 NR\omega\alpha \sin\omega t$$

Q61. A constant and uniform magnetic field $\vec{B} = B_0\hat{k}$ pervades all space. Which one of the following is the correct choice for the vector potential in Coulomb gauge?

(a) $-B_0(x+y)\hat{i}$ (b) $B_0(x+y)\hat{j}$ (c) $B_0 x\hat{j}$ (d) $-\frac{1}{2}B_0\left(x\hat{i} - y\hat{j}\right)$

Ans. : (c)

Solution: Check option (c),

$$\vec{\nabla}\cdot\vec{A} = 0,\ \vec{B} = \vec{\nabla}\times\vec{A} = B_0\hat{k}$$

Q62. A long straight wire, having radius a and resistance per unit length r, carries a current I. The magnitude and direction of the Poynting vector on the surface of the wire is

(a) $I^2 r/2\pi a$, perpendicular to axis of the wire and pointing inwards

(b) $I^2 r/2\pi a$, perpendicular to axis of the wire and pointing outwards

(c) $I^2 r/\pi a$, perpendicular to axis of the wire and pointing inwards

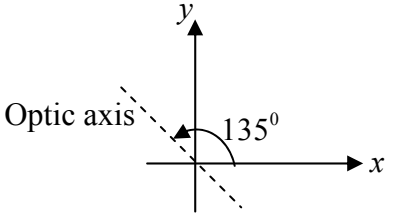
(d) $I^2 r/\pi a$, perpendicular to axis of the wire and pointing outwards

Ans. : (a)

Solution: $\left|\vec{S}\right| = \frac{1}{\mu_0}\left|\left(\vec{E}\times\vec{B}\right)\right| = \frac{1}{\mu_0}\frac{V}{l}\times\frac{\mu_0 I}{2\pi a} = \frac{IR}{l}\times\frac{I}{2\pi a}$

$$\because V = IR,\ r = \frac{R}{l} \Rightarrow \left|\vec{S}\right| = \frac{I^2 r}{2\pi a}$$

Q63. A quarter wave plate introduces a path difference of $\lambda/4$ between the two components of polarization parallel and perpendicular to the optic axis. An electromagnetic wave with $\vec{E} = \left(\hat{x}+\hat{y}\right)E_0 e^{i(kz-\omega t)}$ is incident normally on a quarter wave plate which has its optic axis making an angle 135^0 with the x - axis as shown.



The emergent electromagnetic wave would be

(a) elliptically polarized

(b) circularly polarized

(c) linearly polarized with polarization as that of incident wave

(d) linearly polarized but with polarization at 90^0 to that of the incident wave

Ans. : (c)

Q64. An electromagnetic plane wave is propagating with an intensity $I = 1.0\times10^{5}\,Wm^{-2}$ in a medium with $\in = 3\in_0$ and $\mu = \mu_0$. The amplitude of the electric field inside the medium is _____$\times10^{3}\,Vm^{-1}$ (up to one decimal place).

($\in_0 = 8.85\times10^{-12}\,C^2N^{-1}m^{-2}$, $\mu_0 = 4\pi\times10^{-7}\,NA^{-2}$, $c = 3\times10^{8}\,ms^{-1}$)

Ans. : 6.6

Solution: $I = \frac{1}{2}v\in E^2 \Rightarrow E^2 = \frac{2I}{v\in} = \frac{2I}{\frac{1}{\sqrt{\mu\in}}\in} = 2I\sqrt{\frac{\mu}{\in}}$

$\Rightarrow E^2 = 2\times10^{5}\sqrt{\frac{\mu_0}{3\in_0}} = 2\times10^{5}\sqrt{\frac{4\pi\times10^{-7}}{3\times8.8\times10^{-12}}} \approx 4363.4\times10^{4}$

$\Rightarrow E \approx 66\times10^{2} \approx 6.6\times10^{3}\,V/m$

GATE-2019

Q65. The electric field of an electromagnetic wave is given by $\vec{E} = 3\sin(kz-\omega t)\hat{x} + 4\cos(kz-\omega t)\hat{y}$. The wave is

(a) linearly polarized at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ from the x - axis

(b) linearly polarized at an angle $\tan^{-1}\left(\frac{3}{4}\right)$ from the x - axis

(c) elliptically polarized in clockwise direction when seen travelling towards the observer

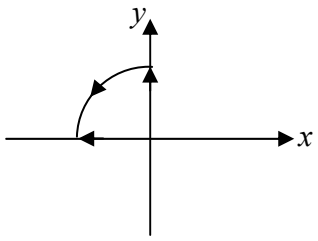
(d) elliptically polarized in counter-clockwise direction when seen travelling towards the observer

Ans. : (d)

Solution: At $z = 0$, $E_x = -3\sin\omega t$, $E_y = 4\cos\omega t$

At $\omega t = 0$, $E_x = 0$, $E_y = 4$

At $\omega t = \frac{\pi}{2}$, $E_x = -3$, $E_y = 0$



Q66. An infinitely long thin cylindrical shell has its axis coinciding with the z-axis. It carries a surface charge density $\sigma_0 \cos\phi$, where ϕ is the polar and σ_0 is a constant. The magnitude of the electric field inside the cylinder is

(a) 0 (b) $\frac{\sigma_0}{2\in_0}$ (c) $\frac{\sigma_0}{3\in_0}$ (d) $\frac{\sigma_0}{4\in_0}$

Ans. : (b)

Solution: $dE = \frac{d\lambda}{2\pi\in_0 R} = \frac{(\sigma_0\cos\phi)(Rd\phi)}{2\pi\in_0 R} = \frac{\sigma_0\cos\phi}{2\pi\in_0}$

Along axis of cylinder $dE_x = dE\cos\phi \Rightarrow E_x = \frac{\sigma_0}{2\pi\in_0}\int_0^{2\pi}\cos^2\phi\, d\phi = \frac{\sigma_0}{2\in_0}$

Q67. A circular loop made of a thin wire has radius $2cm$ and resistance 2Ω. It is placed perpendicular to a uniform magnetic field of magnitude $|\vec{B}_0| = 0.01$ Tesla. At time $t = 0$ the field starts decaying as $\vec{B} = \vec{B}_0 e^{-t/t_0}$, where $t_0 = 1s$. The total charge that passes through a cross section of the wire during the decay is Q. The value of Q in μC (rounded off to two decimal places) is_____

Ans. : 6.28

Solution: $\varepsilon = -\frac{d\phi}{dt} = -\frac{AdB}{dt}$, $I = \frac{\varepsilon}{R} = -\frac{d\phi}{dt}\frac{1}{R}$

$\Rightarrow -\frac{d\phi}{dt} = -\pi r^2 \frac{d}{dt}\left(B_0 e^{-t/t_0}\right) = \pi r^2 B_0 e^{-t}\ (t_0 = 1)$

$Q = \int_0^\infty I(t)\,dt = \int_0^\infty \frac{\pi r^2}{R} B_0 e^{-t} dt = \frac{\pi r^2 B_0}{R}\left|\frac{e^{-t}}{-1}\right|_0^\infty$

$= 3.14\times\left(2\times10^{-2}\right)^2 \times 0.01 = 6.28\mu C$

Q68. The electric field of an electromagnetic wave in vacuum is given by

$$\vec{E} = E_0\cos\left(3y + 4z - 1.5\times10^9 t\right)\hat{x}$$

The wave is reflected from the $z = 0$ surface. If the pressure exerted on the surface is $\alpha \in E_0^2$, the value of α (rounded off to one decimal place) is_____

Ans. : 0.8

Solution: $\vec{K} = 3\hat{y} + 4\hat{z}$

$$\tan\theta_R = \frac{K_y}{K_z} = \frac{3}{4}$$

$$P = 2\frac{I}{c}\cos\theta_R = \frac{2}{c}\times\frac{1}{2}\in_0 cE_0^2 \times\frac{4}{5}$$

$$P = 0.8\in_0 E_0^2$$

Q69. A solid cylinder of radius R has total charge Q distributed uniformly over its volume. It is rotating about its axis with angular speed ω. The magnitude of the total magnetic moment of the cylinder is

(a) $QR^2\omega$ (b) $\frac{1}{2}QR^2\omega$ (c) $\frac{1}{4}QR^2\omega$ (d) $\frac{1}{8}QR^2\omega$

Ans. : (c)

Solution: Magnetic moment due to disc $\mu = \frac{\pi\sigma\omega R^4}{4}$

Due to cylinder $d\mu = \frac{\pi\omega R^4}{4}(\rho dz)$ $\quad(\sigma \to \rho dz)$

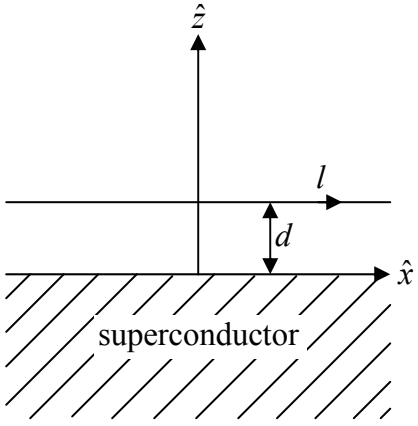
$$\mu = \frac{\pi\omega R^4}{4}\int_0^L \frac{Q}{\pi R^2 L}dz = \frac{Q\omega R^4}{4}$$

Q70. An infinitely long wire parallel to the x-axis is kept at $z = d$ and carries a current I in the positive x direction above a superconductor filling the region $z \le 0$ (see figure). The magnetic field $\vec{B}$ inside the superconductor is zero so that the field just outside the superconductor is parallel to its surface. The magnetic field due to this configuration at a point $(x, y, z > 0)$ is

(a) $\left(\frac{\mu_0 I}{2\pi}\right)\frac{-(z-d)\hat{j} + y\hat{k}}{\left[y^2 + (z-d)^2\right]}$

(b) $\left(\frac{\mu_0 I}{2\pi}\right)\left[\frac{-(z-d)\hat{j} + y\hat{k}}{y^2 + (z-d)^2} + \frac{(z+d)\hat{j} - y\hat{k}}{y^2 + (z+d)^2}\right]$

(c) $\left(\frac{\mu_0 I}{2\pi}\right)\left[\frac{-(z-d)\hat{j} + y\hat{k}}{y^2 + (z-d)^2} - \frac{(z+d)\hat{j} - y\hat{k}}{y^2 + (z+d)^2}\right]$



(d) $\left(\frac{\mu_0 I}{2\pi}\right)\left[\frac{y\hat{j}+(z-d)\hat{k}}{y^2+(z-d)^2}+\frac{y\hat{j}-(z+d)\hat{k}}{y^2+(z+d)^2}\right]$

Ans. : (b)

Solution: Verify that $\vec{B}=0$, when $d=0$

Q71. The vector potential inside a long solenoid with n turns per unit length and carrying current I, written in cylindrical coordinates is $\vec{A}(s,\phi,z)=\frac{\mu_0 nI}{2}s\hat{\phi}$. If the term $\frac{\mu_0 nI}{2}s\left(\alpha\cos\phi\hat{\phi}+\beta\sin\phi\hat{s}\right)$, where $\alpha\neq 0,\ \beta\neq 0$ is added to $\vec{A}(S,\phi,z)$, the magnetic field remains the same if

(a) $\alpha=\beta$ (b) $\alpha=-\beta$ (c) $\alpha=2\beta$ (d) $\alpha=\frac{\beta}{2}$

Useful formulae: $\vec{\Delta}t=\frac{\partial t}{\partial S}\hat{S}+\frac{1}{S}\frac{\partial t}{\partial\phi}\hat{\phi}+\frac{\partial t}{\partial z}\hat{z};$

$\vec{\nabla}\times\vec{v}=\left(\frac{1}{s}\frac{\partial v_z}{\partial\phi}-\frac{\partial v_\phi}{\partial z}\right)\hat{s}+\left(\frac{\partial v_s}{\partial z}-\frac{\partial v_z}{\partial s}\hat{\phi}+\frac{1}{s}\left(\frac{\partial(sv_\phi)}{\partial s}-\frac{\partial v_s}{\partial\phi}\right)\hat{z}\right)$

Ans. : (d)

Solution: $\vec{B}=\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & 0\end{vmatrix}=\mu_0 nI\hat{z}$

$\vec{B}'=\vec{\nabla}\times\vec{A}'=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & 0\end{vmatrix}=\mu_0 nI\left[(\alpha\cos\phi+1)-\frac{\beta\cos\phi}{2}\right]\hat{z}$

Equate $\vec{B}'=\vec{B}\Rightarrow\left[(\alpha\cos\phi+1)-\frac{\beta\cos\phi}{2}\right]=\mu_0 nI$

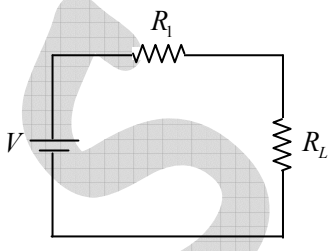
$\Rightarrow\alpha\cos\phi=\frac{\beta}{2}\cos\phi\Rightarrow\alpha=\frac{\beta}{2}$

Q72. For a given load resistance $R_L = 4.7$ ohm, the power transfer efficiencies $\left(\eta = \frac{P_{load}}{P_{total}}\right)$ of a dc voltage source and a dc current source with internal resistances R_1 and R_2, respectively, are equal. The product R_1R_2 in units of ohm^2 (rounded off to one decimal place) is_____

Ans. : 22.09

Solution: For dc voltage source

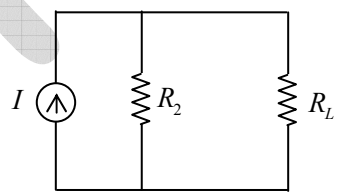
$$P_{total} = \frac{V^2}{R_1 + R_L} \text{ and } P_{R_L} = \left(\frac{V}{R_1 + R_L}\right)^2 R_L$$



$$\eta_{dc\ vol} = \frac{P_{R_L}}{P_{total}} = \frac{R_L}{R_1 + R_L}$$

For dc current source

$$P_{total} = I^2\left(\frac{R_2R_L}{R_2 + R_L}\right) \text{ and } P_{R_L} = I_L^2R_L = \left(\frac{R_2I}{R_2 + R_L}\right)^2 R_L$$



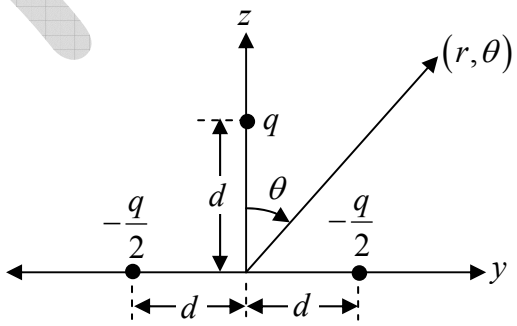
$$\eta_{dc\ curr} = \frac{P_{R_L}}{P_{total}} = \frac{R_2}{R_2 + R_L}$$

Since $\eta_{dc\ vol} = \eta_{dc\ curr}$

$$\Rightarrow \frac{R_L}{R_1 + R_L} = \frac{R_2}{R_2 + R_L} \Rightarrow R_L(R_2 + R_L) = R_2(R_1 + R_L) \Rightarrow R_1R_2 = R_L^2$$

$$\Rightarrow R_1R_2 = (4.7)^2 = 22.09\ \Omega^2$$

Q73. Consider a system of three charges as shown in the figure below:



For $r = 10\ m; \theta = 60$ degrees; $q = 10^{-6}$ Coulomb, and $d = 10^{-3}\ m$, the electric dipole potential in volts (rounded off to three decimal places) at a point (r, θ) is _____

[Use: $\frac{1}{4\pi \in_0} = 9 \times 10^9 \frac{Nm^2}{C^2}$]

Ans. : 0.045

Solution: Monopole moment $= -\frac{q}{2} - \frac{q}{2} + q = 0$

$$\vec{p} = -\frac{q}{2} \times (-d\hat{y}) - \frac{q}{2}(d\hat{y}) + q(d\hat{z})$$

$$\vec{p} = qd\hat{z}$$

$$V(r,\theta) = \frac{1}{4\pi \in_0} \frac{\vec{p} \cdot r}{r^2} = \frac{1}{4\pi \in_0} \frac{qd\cos\theta}{r^2}$$

$$V(r,\theta) = 9 \times 10^9 \times \frac{10^{-6} \times 10^{-3} \times \cos 60^0}{(10)^2}$$

$$= 9 \times 10^9 \times \frac{10^{-9}}{2 \times 100} = 0.045$$

Q74. The electric field of an electromagnetic wave is given by $\vec{E} = 3\sin(kz - \omega t)\hat{x} + 4\cos(kz - \omega t)\hat{y}$. The wave is

(a) linearly polarized at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ from the x - axis

(b) linearly polarized at an angle $\tan^{-1}\left(\frac{3}{4}\right)$ from the x - axis

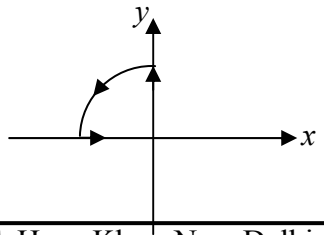
(c) elliptically polarized in clockwise direction when seen travelling towards the observer

(d) elliptically polarized in counter-clockwise direction when seen travelling towards the observer

Ans. : (d)

Solution: At $z = 0,\ E_x = -3\sin\omega t,\ E_y = 4\cos\omega t$

At $\omega t = 0,\ E_x = 0,\ E_y = 4$



At $\omega t = \frac{\pi}{2},\ E_x = -3,\ E_y = 0$

Q75. In a set of N successive polarizers, the m^{th} polarizer makes an angle $\left(\frac{m\pi}{2N}\right)$ with the vertical. A vertically polarized light beam of intensity I_0 is incident on two such sets with $N = N_1$ and $N = N_2$, where $N_2 > N_1$. Let the intensity of light beams coming out be $I(N_1)$ and $I(N_2)$, respectively. Which of the following statements is correct about the two outgoing beams?

(a) $I(N_2) > I(N_1)$; the polarization in each case is vertical

(b) $I(N_2) < I(N_1)$; the polarization in each case is vertical

(c) $I(N_2) > I(N_1)$; the polarization in each case is horizontal

(d) $I(N_2) < I(N_1)$; the polarization in each case is horizontal

Ans. : (c)

Solution: $I(N_1) = I_0\left[\cos\left(\frac{n/2}{N_1}\right)\right]^{2N_1}$, $I(N_2) = I_0\left[\cos\left(\frac{n/2}{N_2}\right)\right]^{2N_2}$

$I(N_2) > I(N_1)$

For last polarization, pass axis will be horizontal.

Ex: $N_1 = 5$

$I(5) = I_0\left[\cos\left(18^*\right)\right]^{10} = 0.605\, I_0$

$N_2 = 10$

$I(10) = I_0\left[\cos\left(9^*\right)\right]^{20} = 0.780\, I_0$

$I(10) > I(5)$

QUANTUM MECHANICS SOLUTIONS

GATE- 2010

Q1. Which of the following is an allowed wavefunction for a particle in a bound state? N is a constant and $\alpha, \beta > 0$.

(a) $\psi = N\dfrac{e^{-\alpha r}}{r^3}$ (b) $\psi = N\left(1 - e^{-\alpha r}\right)$

(c) $\psi = Ne^{-\alpha x}e^{-\beta\left(x^2+y^2+z^2\right)}$ (d) $\psi = \begin{cases} \text{non - zero constant} & \text{if } r < R \\ 0 & \text{if } r > R \end{cases}$

Ans: (c)

Q2. A particle of mass m is confined in the potential

$$V(x) = \begin{cases} \dfrac{1}{2}m\omega^2 x^2, & \text{for } x > 0 \\ \infty \quad , & \text{for } x \le 0 \end{cases}$$

Let the wavefunction of the particle be given by

$$\psi(x) = -\frac{1}{\sqrt{5}}\psi_0 + \frac{2}{\sqrt{5}}\psi_1,$$

where ψ_0 and ψ_1 are the eigenfunctions of the ground state and the first excited state respectively. The expectation value of the energy is

(a) $\dfrac{31}{10}\hbar\omega$ (b) $\dfrac{25}{10}\hbar\omega$ (c) $\dfrac{13}{10}\hbar\omega$ (d) $\dfrac{11}{10}\hbar\omega$

Ans: (a)

Solution: For half parabolic potential

$$E_0 = \frac{3}{2}\hbar\omega, E_1 = \frac{7}{2}\hbar\omega \Rightarrow \langle E\rangle = \left(\frac{1}{5}\times\frac{3}{2} + \frac{4}{5}\times\frac{7}{2}\right)\hbar\omega = \frac{31}{10}\hbar\omega.$$

Q3. For a spin-s particle, in the eigen basis of $\vec{S}^2$, S_x the expectation value $\langle sm|S_x^2|sm\rangle$ is

(a) $\dfrac{\hbar^2\left\{s(s+1) - m^2\right\}}{2}$ (b) $\hbar^2\left\{s(s+1) - 2m^2\right\}$

(c) $\hbar^2\left\{s(s+1) - m^2\right\}$ (d) $\hbar^2 m^2$

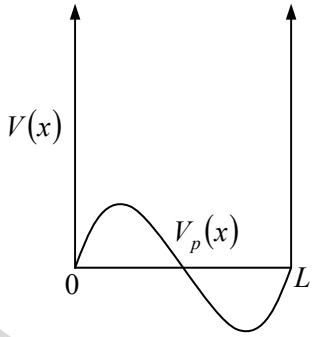
Ans: (a)

Solution: $\langle sm|S_x^2|sm\rangle = \dfrac{1}{4}\langle sm|\left(S_+ + S_-\right)^2|sm\rangle = \dfrac{1}{4}\langle sm|S_+^2 + S_-^2 + S_+S_- + S_-S_+|sm\rangle$

$$= \frac{1}{4}\langle sm|S_+S_- + S_-S_+|sm\rangle = \frac{\hbar^2}{2}\left[s(s+1) - m^2\right] \qquad \left[\because S_+S_- + S_-S_+ = 2\left(S^2 - S_z^2\right)\right]$$

Q4. A particle of mass m is confined in an infinite potential well:

$$V(x)=\begin{cases}0, & \text{if } 0<x<L,\\ \infty, & \text{otherwise.}\end{cases}$$



It is subjected to a perturbing potential $V_p(x)=V_o\sin\left(\frac{2\pi x}{L}\right)$ within the well. Let $E^{(1)}$ and $E^{(2)}$ be corrections to the ground state energy in the first and second order in V_0, respectively. Which of the following are true?

(a) $E^{(1)}=0; E^{(2)}<0$ (b) $E^{(1)}>0; E^{(2)}=0$

(c) $E^{(1)}=0; E^{(2)}$ depends on the sign of V_0 (d) $E^{(1)}<0; E^{(2)}<0$

Ans: (a)

Solution: $E_1^1=\frac{2}{L}\int_0^L V_0\sin\frac{2\pi x}{L}dx\ =0$; $E_1^2=\sum_{m\neq 1}\frac{\left|\langle\psi_m|V_P|\psi_1\rangle\right|^2}{E_1-E_m}$ $\because E_1<E_m$ so $E_1^2=-ve$.

GATE- 2011

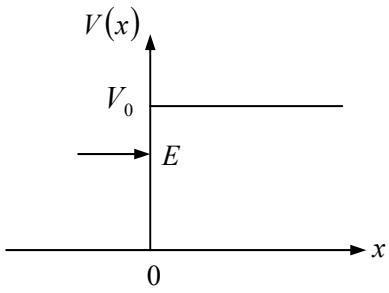
Q5. The quantum mechanical operator for the momentum of a particle moving in one dimension is given by

(a) $i\hbar\frac{d}{dx}$ (b) $-i\hbar\frac{d}{dx}$ (c) $i\hbar\frac{\partial}{\partial t}$ (d) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

Ans: (b)

Q6. An electron with energy E is incident from left on a potential barrier, given by

$$V(x)=\begin{cases}0, & \text{for } x<0\\ V_0, & \text{for } x>0\end{cases}$$



as shown in the figure. For $E < V_0$, the space part of the wavefunction for $x > 0$ is of the form

(a) e^{ax} (b) e^{-ax} (c) e^{iax} (d) e^{-iax}

Ans: (b)

Solution: $\because E<V_0$, so there is decaying wave function.

Q7. If L_x, L_y and L_z are respectively the x, y and z components of angular momentum operator L. The commutator $[L_x L_y, L_z]$ is equal to

(a) $i\hbar\left(L_x^2 + L_y^2\right)$ (b) $2i\hbar L_z$ (c) $i\hbar\left(L_x^2 - L_y^2\right)$ (d) 0

Ans: (c)

Solution: $\left[L_x L_y, L_z\right] = L_x\left[L_y L_z\right] + \left[L_x, L_z\right]L_y = i\hbar\left(L_x^2 - L_y^2\right)$

Q8. The normalized ground state wavefunciton of a hydrogen atom is given by $\psi(r) = \frac{1}{\sqrt{4\pi}}\frac{2}{a^{3/2}}e^{-r/a}$, where a is the Bohr radius and r is the distance of the electron from the nucleus, located at the origin. The expectation value $\left\langle \frac{1}{r^2} \right\rangle$ is

(a) $\frac{8\pi}{a^2}$ (b) $\frac{4\pi}{a^2}$ (c) $\frac{4}{a^2}$ (d) $\frac{2}{a^2}$

Ans: (d)

Solution: $\left\langle \frac{1}{r^2} \right\rangle = \frac{4}{4\pi a^3}\int_0^\infty \frac{1}{r^2} r^2 e^{-\frac{2r}{a}} dr \int_0^\pi \int_0^{2\pi} \sin\theta \; d\theta \; d\phi = \frac{2}{a^2}$

Q9. The normalized eigenstates of a particle in a one-dimensional potential well

$$V(x) = \begin{cases} 0 & \text{if } 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

are given by $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$, where n = 1, 2, 3,….

The particle is subjected to a perturbation

$$V'(x) = \begin{cases} V_0 \cos\left(\frac{\pi x}{a}\right), & \text{for } 0 \le x \le \frac{a}{2} \\ 0 \quad , & \text{otherwise} \end{cases}$$

The shift in the ground state energy due to the perturbation, in the first order perturbation theory,

(a) $\frac{2V_o}{3\pi}$ (b) $\frac{V_o}{3\pi}$ (c) $-\frac{V_o}{3\pi}$ (d) $-\frac{2V_o}{3\pi}$

Ans: (a)

Solution: $E_1^1 = \int_0^{a/2} \psi_1^* V'(x) \psi_1 dx = \frac{2}{a}\int_0^{a/2} \sin^2\left(\frac{\pi x}{a}\right) V_0 \cos\left(\frac{\pi x}{a}\right) dx = \frac{2}{a} V_0 \frac{\sin^3\frac{\pi x}{a}}{3\frac{\pi}{a}}\Bigg|_0^{a/2} = \frac{2V_0}{3\pi}$

Common data questions Q10 and Q11

In a one-dimensional harmonic oscillator, φ_0, φ_1 and φ_2 are respectively the ground, first and the second excited states. These three states are normalized and are orthogonal to one another ψ_1 and ψ_2 are two states defined by

$$\psi_1 = \varphi_0 - 2\varphi_1 + 3\varphi_2,\ \psi_2 = \varphi_0 - \varphi_1 + \alpha\varphi_2,\ \psi_2 = \varphi_0 - \varphi_1 + \alpha\varphi_2$$

where α is a constant

Q10. The value of α which ψ_2 is orthogonal to ψ_1 is

(a) 2 (b) 1 (c) – 1 (d) – 2

Ans: (c)

Solution: For orthogonal condition scalar product $(\psi_2, \psi_1) = 0$, so $1 + 2 + 3\alpha = 0 \Rightarrow \alpha = -1$

Q11. For the value of α determined in Q10, the expectation value of energy of the oscillator in the state ψ_2 is

(a) $\hbar\omega$ (b) $3\hbar\omega/2$ (c) $3\hbar\omega$ (d) $9\hbar\omega/2$

Ans: (b)

Solution: $\psi_2 = \phi_0 - \phi_1 + \alpha\phi_2$ put $\alpha = -1, \langle H \rangle = \dfrac{\langle \psi_2 | H | \psi_2 \rangle}{\langle \psi_2 | \psi_2 \rangle} = \dfrac{\frac{\hbar\omega}{2} + \frac{3\hbar\omega}{2} + \frac{5\hbar\omega}{2}}{3} = \dfrac{3}{2}\hbar\omega$

GATE- 2012

Q12. A particle of mass m is confined in a two dimensional square well potential of dimension a. This potential $V(x, y)$ is given by

$V(x, y) = 0$ for $-a < x < a$ and $-a < y < a$

$= \infty$ elsewhere

The energy of the first excited state for this particle is given by,

(a) $\dfrac{\pi^2\hbar^2}{ma^2}$ (b) $\dfrac{2\pi^2\hbar^2}{ma^2}$ (c) $\dfrac{5\pi^2\hbar^2}{8ma^2}$ (d) $\dfrac{4\pi^2\hbar^2}{ma^2}$

Ans: (c)

Solution: $E = \left(n_x^2 + n_y^2\right)\dfrac{\pi^2\hbar^2}{2m(2a)^2} = \left(n_x^2 + n_y^2\right)\dfrac{\pi^2\hbar^2}{8ma^2} = \dfrac{5\pi^2\hbar^2}{8ma^2}$ $\because n_x = 1, n_y = 2$.

Q13. Consider the wavefunction $\psi = \psi(\vec{r}_1, \vec{r}_2)\chi_s$ for a fermionic system consisting of two spin-half particles. The spatial part of the wavefunction is given by

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}[\phi_1(\vec{r}_1)\phi_2(\vec{r}_2) + \phi_2(\vec{r}_1)\phi_1(\vec{r}_2)]$$

where ϕ_1 and ϕ_2 are single particle states. The spin part χ_s of the wavefunction with spin states $\alpha(+1/2)$ and $\beta(-1/2)$ should be

(a) $\frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha)$ (b) $\frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha)$ (c) $\alpha\alpha$ (d) $\beta\beta$

Ans: (b)

Solution: Since $\psi(r_1, r_2)$ is symmetric the total wavefunction must be antisymmetric for fermions so spin part must be antisymmetric.

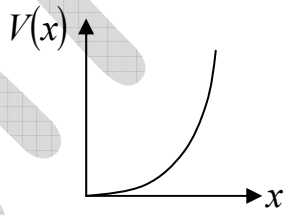
Q14. A particle is constrained to move in a truncated harmonic potential well ($x > 0$) as shown in the figure. Which one of the following statements is CORRECT?

(a) The parity of the first excited state is even

(b) The parity of the ground state is even

(c) the ground state energy is $\frac{1}{2}\hbar\omega$

(d) The first excited state energy is $\frac{7}{2}\hbar\omega$



Ans: (d)

Solution: There is only odd parity. Ground state is $\frac{3}{2}\hbar\omega$ and first excited $= \frac{7}{2}\hbar\omega$

Q15. Consider a system in the unperturbed state described by the Hamiltonian, $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The system is subjected to a perturbation of the form $H' = \begin{pmatrix} \delta & \delta \\ \delta & \delta \end{pmatrix}$, where $\delta << 1$. The energy eigenvalues of the perturbed system using the first order perturbation approximation are

(a) 1 and $(1 + 2\delta)$ (b) $(1 + \delta)$ and $(1 - \delta)$

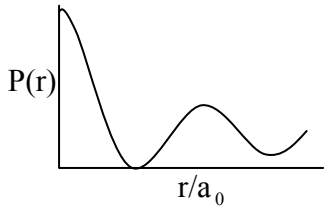
(c) $(1+ 2\delta)$ and $(1 - 2\delta)$ (d) $(1+ \delta)$ and $(1 - 2\delta)$

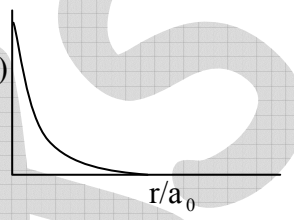
Ans: (a)

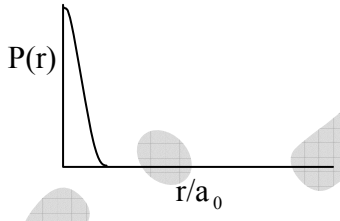
Solution: $H_0 + H'$, H_0 is degenerate so after using degenerate perturbation through diagonalized H' one will get $H' = \delta\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

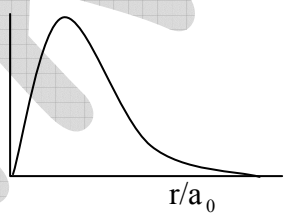
So $E = 1 + 2\delta$ and $1 + 0\delta$.

Q16. The ground state wavefunction for the hydrogen atom is given by $\psi_{100} = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$, where a_0 is the Bohr radius. The plot of the radial probability density, $P(r)$ for the hydrogen atom in the ground state is

(a) 

(b) 

(c) 

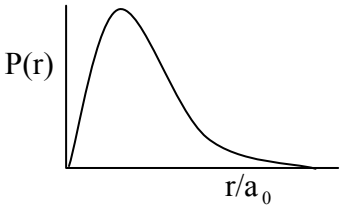
(d) 

Ans: (d)

Solution: The ground state is given by $\psi_{100} = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$

Radial probability function

$P(r) = |\psi|^2 r^2 = \frac{1}{4\pi}\frac{1}{a_0^3} r^2 e^{-2r/a_0}$



Common Data for Questions 17–18

The wavefunction of particle moving in free space is given by, $\psi = \left(e^{ikx} + 2e^{-ikx}\right)$

Q17. The energy of the particle is

(a) $\frac{5\hbar^2 k^2}{2m}$ (b) $\frac{3\hbar^2 k^2}{4m}$ (c) $\frac{\hbar^2 k^2}{2m}$ (d) $\frac{\hbar^2 k^2}{m}$

Ans: (c)

Solution: $H\psi = E\psi$, $H\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{-\hbar^2}{2m}\left[(ik)(ik)e^{ikx} + 2(-ik)(-ik)e^{-ikx}\right]$

$$\Rightarrow H\psi = \frac{\hbar^2k^2}{2m}(e^{ikx} + 2e^{-2ikx}) = \frac{\hbar^2k^2}{2m}\psi$$

Q18. The probability current density for the real part of the wavefunction is

(a) 1 (b) $\frac{\hbar k}{m}$ (c) $\frac{\hbar k}{2m}$ (d) 0

Ans: (d)

Solution: The real part of the wave function $\psi_{real} = \cos kx + 2\cos kx$

Current density for real part of wave function = 0

GATE- 2013

Q19. Which one of the following commutation relations is **NOT CORRECT**? Here, symbols have their usual meanings.

(a) $\left[L^2, L_z\right] = 0$ (b) $\left[L_x, L_y\right] = i\hbar L_z$

(c) $\left[L_z, L_+\right] = \hbar L_+$ (d) $\left[L_z, L_-\right] = \hbar L_-$

Ans: (d)

Q20. A proton is confined to a cubic box, whose sides have length $10^{-12}\,m$. What is the minimum kinetic energy of the proton? The mass of proton is $1.67\times10^{-27}\,kg$ and Planck's constant is $6.63\times10^{-34}\,Js$.

(a) $1.1\times10^{-17}\,J$ (b) $3.3\times10^{-17}\,J$ (c) $9.9\times10^{-17}\,J$ (d) $6.6\times10^{-17}\,J$

Ans: (c)

Solution: $\frac{3\pi^2\hbar^2}{2ma^2} = 9.9\times10^{-17}$

Q21. A spin-half particle is in a linear superposition $0.8\left|\uparrow\right\rangle + 0.6\left|\downarrow\right\rangle$ of its spin-up and spin-down states. If $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ are the eigenstates of σ_z, then what is the expectation value up to one decimal place, of the operator $10\sigma_z + 5\sigma_x$? Here, symbols have their usual meanings. _____

Ans: 7.6

Solution: $|\psi\rangle = .8|\uparrow\rangle + .6|\downarrow\rangle = 0.8\begin{pmatrix}1\\0\end{pmatrix} + 0.6\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0.8\\0.6\end{pmatrix}$

Operator $A = 10\sigma_z + 5\sigma_x = 10\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} + 5\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \Rightarrow A = \begin{pmatrix}10 & 5\\5 & -10\end{pmatrix}$

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \begin{pmatrix}0.8 & 0.6\end{pmatrix}\begin{pmatrix}10 & 5\\5 & -10\end{pmatrix}\begin{pmatrix}0.8\\0.6\end{pmatrix} = (8.8 - 1.2) = 7.6$$

Q22. Consider the wave function $Ae^{ikr}(r_0/r)$, where A is the normalization constant. For $r = 2r_0$, the magnitude of probability current density up to two decimal places, in units of $(A^2\hbar k/m)$ is _____

Ans: 0.25

Solution: $\vec{J} = |\psi|^2\frac{\hbar k}{m} = |A|^2\left|\frac{r_0}{r}\right|^2\frac{\hbar k}{m} \Rightarrow J = |A|^2\left|\frac{r_0}{2r_0}\right|^2\frac{\hbar k}{m} \Rightarrow J = |A|^2\frac{\hbar k}{4m} = (0.25)|A|^2\frac{\hbar k}{m}$

Common data questions 23 and 24

To the given unperturbed Hamiltonian $\begin{bmatrix}5 & 2 & 0\\2 & 5 & 0\\0 & 0 & 2\end{bmatrix}$

we add a small perturbation given by $\varepsilon\begin{bmatrix}1 & 1 & 1\\1 & 1 & -1\\1 & -1 & 1\end{bmatrix}$ where ε is small quantity.

Q23. The ground state eigenvector of the unperturbed Hamiltonian is

(a) $(1/\sqrt{2}, 1\sqrt{2}, 0)$ (b) $(1/\sqrt{2}, -1/\sqrt{2}, 0)$

(c) $(0,0,1)$ (d) $(1,0,0)$

Ans: (c)

$$H_0 = \begin{bmatrix}5 & 2 & 0\\2 & 5 & 0\\0 & 0 & 2\end{bmatrix},\ H_P = \varepsilon\begin{pmatrix}1 & 1 & 1\\1 & 1 & -1\\1 & -1 & 1\end{pmatrix}$$

Eigen value of H_0 is $E_1 = 2,\ E_2 = 3,\ E_3 = 7$ and the Eigen vector corresponds to $|\phi_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Q24. A pair of eigenvalues of the perturbed Hamiltonian, using first order perturbation theory, is

(a) $3+2\varepsilon, 7+2\varepsilon$ (b) $3+2\varepsilon, +2+\varepsilon$ (c) $3, 7+2\varepsilon$ (d) $3, 2+2\varepsilon$

Ans: (c)

Solution: $E_1' = \langle\phi_1|H_P|\phi_1\rangle = 1\varepsilon \Rightarrow E_1 = 2+1\varepsilon$

$$E_2' = \langle\phi_2|H_P|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 & 0\end{pmatrix}.\varepsilon\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\end{pmatrix} = \in\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\\ 0\end{pmatrix} = 0$$

$$E_3' = \langle\phi_3|H_P|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\end{pmatrix}.\varepsilon\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix} = \varepsilon.\frac{1}{2}\begin{pmatrix}2 & 2 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$$

$\Rightarrow E_3' = \frac{1}{2}(4)\varepsilon = 2\varepsilon$

$E_1 = 2+1\varepsilon,\ E_2 = 3+0\varepsilon,\ E_3 = 7+2\varepsilon$.

GATE- 2014

Q25. The recoil momentum of an atom is p_A when it emits an infrared photon of wavelength $1500\,nm$, and it is p_B when it emits a photon of visible wavelength $500nm$. The ratio $\frac{p_A}{p_B}$ is

(a) 1 : 1 (b) $1:\sqrt{3}$ (c) 1 : 3 (d) 3 : 2

Ans: (c)

Solution: $p = \frac{h}{\lambda}$, $\frac{p_A}{p_B} = \frac{\lambda_B}{\lambda_A}$, $\frac{\lambda_B}{\lambda_A} = \frac{500}{1500} = 1:3$

Q26. The ground state and first excited state wave function of a one dimensional infinite potential well are ψ_1 and ψ_2 respectively. When two spin-up electrons are placed in this

potential which one of the following with x_1 and x_2 denoting the position of the two electrons correctly represents the space part of the ground state wave function of the system?

(a) $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_1)-\psi_1(x_2)\psi_2(x_2)]$ (b) $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2)+\psi_1(x_2)\psi_2(x_1)]$

(c) $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_1)+\psi_1(x_2)\psi_2(x_2)]$ (d $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2)-\psi_1(x_2)\psi_2(x_1)]$)

Ans: (d)

Solution: From the given information only possible spin configuration is symmetric in nature so space part will anti symmetric

$$\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2)-\psi_1(x_2)\psi_2(x_1)]$$

Q27. If $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the spin angular momentum, then $\vec{L}.\vec{S}$ does not commute with

(a) S_z (b) L^2 (c) S^2 (d) $(\vec{L}+\vec{S})^2$

Ans: (d)

Q28. An electron in the ground state of the hydrogen atom has the wave function $\psi(\vec{r})=\frac{1}{\sqrt{\pi a_0^3}}e^{-\left(\frac{r}{a_0}\right)}$, where a_0 is constant. The expectation value of the operator $\hat{Q}=z^2-r^2$, where $z=r\cos\theta$ is (Hint: $\int_0^\infty e^{-ar}r^n dr=\frac{\lfloor n}{a^{n+1}}=\frac{(n-1)!}{a^{n+1}}$)

(a) $\frac{-a_0^2}{2}$ (b) $-a_0^2$ (c) $\frac{-3a_0^2}{2}$ (d) $-2a_0^2$

Ans: (d)

Solution: $\langle\hat{Q}\rangle=\langle z^2\rangle-\langle r^2\rangle \Rightarrow a_0^2-3a_0^2=-2a_0^2$

Q29. A particle of mass m is subjected to a potential

$$V(x,y)=\frac{1}{2}m\omega^2(x^2+y^2), -\infty\le x\le\infty, -\infty\le y\le\infty$$

The state with energy $4\hbar\omega$ is $g-$fold degenerate. The value of g is _____

Ans: 4

Solution: This is two isotropic dimensional harmonic oscillator the energy eigen value for nth state is $E_n=(n+1)\hbar\omega$ with degeneracy $g_n=(n+1)$ so degeneracy for $4\hbar\omega$ is 4.

Q30. A hydrogen atom is in the state

$$\psi=\sqrt{\frac{8}{21}}\psi_{200}-\sqrt{\frac{3}{7}}\psi_{310}+\sqrt{\frac{4}{21}}\psi_{321},$$

where n,l,m in ψ_{nlm} denote the principal, orbital and magnetic quantum numbers, respectively. If $\vec{L}$ is the angular momentum operator, then the average value of L^2 is_____$\hbar^2$

Ans: 2

Solution: If L^2 will measure on state ψ the measurement is $0\hbar^2$, $2\hbar^2$ and $6\hbar^2$ with probability $\frac{8}{21},\frac{3}{7},\frac{4}{21}$ so, $\left\langle L^2\right\rangle=2\hbar^2\times\frac{3}{7}+6\hbar^2\times\frac{4}{21}=2\hbar^2$

Q31. ψ_1 and ψ_2 are two orthogonal states of a spin $\frac{1}{2}$ system. It is given that

$\psi_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{2}{3}}\begin{pmatrix}0\\1\end{pmatrix}$, where $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ represent the spin-up and spin-down states, respectively. When the system is in the state ψ_2 its probability to be in the spin-up state is_____

Ans: $\frac{2}{3}$

Solution: If is $\psi_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{2}{3}}\begin{pmatrix}0\\1\end{pmatrix}$, then $\psi_2=\sqrt{\frac{2}{3}}\begin{pmatrix}1\\0\end{pmatrix}+\sqrt{\frac{1}{3}}\begin{pmatrix}0\\1\end{pmatrix}$,

so probability that ψ_2 is in up state is $\frac{2}{3}$

Q32. A particle is confined to a one dimensional potential box, with the potential

$$V(x)=\begin{cases}0, & 0<x<a\\ \infty, & \text{otherwise}\end{cases}$$

If particle is subjected to a perturbation within the box. $W = \beta x$. Where β is small constant, the first order correction to the ground state energy is

(a) 0 (b) $a\beta/4$ (c) $a\beta/2$ (d) $a\beta$

Ans: (c)

Solution: First order energy correction is $\langle W \rangle = \beta \langle x \rangle$. The average value of position in ground state is $\langle x \rangle = \frac{a}{2}$ so answer is $a\beta/2$

Q33. A one dimensional harmonic oscillator is in the superposition of number state $|n\rangle$ given by $|\psi\rangle = \frac{1}{2}|2\rangle + \frac{\sqrt{3}}{2}|3\rangle$.

The average energy of the oscillator in the given state is_____$\hbar\omega$.

Ans: 3.25

Solution: Average energy will $\frac{\frac{1}{4}\cdot\frac{5\hbar\omega}{2} + \frac{3}{4}\cdot\frac{7\hbar\omega}{2}}{\frac{1}{4}+\frac{3}{4}} = 3.25\hbar\omega$

Q34. If L_+ and L_- are the angular momentum ladder operators then the expectation value of $(L_+L_- + L_-L_+)$ in the state $|l=1, m=1\rangle$ of an atom is _____$\hbar^2$

Ans: 2

Solution: $(L_+L_- + L_-L_+) = 2(L^2 - L_z^2) = 2(l.(l+1) - m^2)\hbar^2 = 2\hbar^2$

GATE- 2015

Q35. An operator for a spin $\frac{1}{2}$ particle is given by $\hat{A} = \lambda\vec{\sigma}\cdot\vec{B}$, where $\vec{B} = \frac{B}{\sqrt{2}}(\hat{x}+\hat{y})$, $\vec{\sigma}$ denotes Pauli matrices and λ is a constant. The eigenvalues of $\hat{A}$ are

(a) $\pm\frac{\lambda B}{\sqrt{2}}$ (b) $\pm\lambda B$ (c) $0, \lambda B$ (d) $0, -\lambda B$

Ans.: (b)

Solution: $\hat{A} = \lambda\vec{\sigma}\cdot\vec{\beta}, \quad \vec{B} = \frac{B}{\sqrt{2}}(\hat{x}+\hat{y})$

$$\hat{A} = \lambda\left(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z\right) \Rightarrow \hat{A} = \lambda\left[\sigma_x B_x + \sigma_y B_y\right]$$

$$\hat{A} = \lambda\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{B}{\sqrt{2}} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{B}{\sqrt{2}}\right] \Rightarrow \hat{A} = \frac{\lambda B}{\sqrt{2}}\begin{bmatrix}0 & 1-i\\ 1+i & 0\end{bmatrix}$$

$$|A - \lambda' I| = 0 \Rightarrow \frac{\lambda B}{\sqrt{2}}\begin{bmatrix}-\lambda & 1-i\\ 1+i & -\lambda\end{bmatrix} = 0 \Rightarrow \lambda' = \pm\lambda B$$

Q36. The Pauli matrices for three spin $-\frac{1}{2}$ particles are $\vec{\sigma}_1, \vec{\sigma}_2$ and $\vec{\sigma}_3$, respectively. The dimension of the Hilbert space required to define an operator $\hat{O} = \vec{\sigma}_1\cdot\vec{\sigma}_2\times\vec{\sigma}_3$ is_____

Ans.: 8

Solution: $\sigma_2\times\sigma_3$ has dimension of 4 and $\sigma_1.\ \sigma_2\times\sigma_3$ has dimension of $2\times 4 = 8$

Q37. Let $\vec{L}$ and $\vec{p}$ be the angular and linear momentum operators, respectively, for a a particle. The commutator $\left[L_x, p_y\right]$ gives

(a) $-i\hbar p_z$ (b) 0 (c) $i\hbar p_x$ (d) $i\hbar p_z$

Ans.: (d)

Solution: $\left[L_x, p_y\right] = \left[yp_z - zp_y, p_y\right] = \left[yp_z, p_y\right] - \left[zp_y, p_y\right] = \left[y, p_y\right]p_z$

$\because \left[p_y, p_y\right] = 0$ and $\left[z, p_y\right] = 0 \Rightarrow \left[L_x, p_y\right] = i\hbar p_z \qquad \because \left[y, p_y\right] = i\hbar$

Q38. Consider a system of eight non-interacting, identical quantum particles of spin $-\frac{3}{2}$ in a one dimensional box of length L. The minimum excitation energy of the system, in units of $\frac{\pi^2\hbar^2}{2mL^2}$ is _____

Ans.: 5

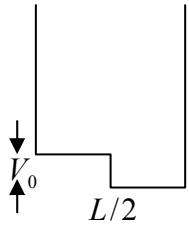
Solution: spin $\frac{3}{2} \Rightarrow$ degeneracy $= (2S+1) = \left(2\times\frac{3}{2}+1\right) = 4$

$$E_{ground} = 4\times\frac{\pi^2\hbar^2}{2mL^2} + 4\times\frac{4\pi^2\hbar^2}{2mL^2} = \frac{20\pi^2\hbar^2}{2mL^2}$$

$$E_{excited}^{I^{st}} = 4\times\frac{\pi^2\hbar^2}{2mL^2} + 3\times4\frac{\pi^2\hbar^2}{2mL^2} + 1\times9\frac{\pi^2\hbar^2}{2mL^2} = 25\frac{\pi^2\hbar^2}{2mL^2}$$

Now minimum excitation energy $\Delta E = E_{excited}^{I^{st}} - E_{ground} = 25\frac{\pi^2\hbar^2}{2mL^2} - 20\frac{\pi^2\hbar^2}{2mL^2} = 5\frac{\pi^2\hbar^2}{2mL^2}$

Q39. A particle is confined in a box of length L as shown in the figure. If the potential V_0 is treated as a perturbation, including the first order correction, the ground state energy is



(a) $E = \frac{\hbar^2\pi^2}{2mL^2} + V_0$ (b) $E = \frac{\hbar^2\pi^2}{2mL^2} - \frac{V_0}{2}$

(c) $E = \frac{\hbar^2\pi^2}{2mL^2} + \frac{V_0}{4}$ (d) $E = \frac{\hbar^2\pi^2}{2mL^2} + \frac{V_0}{2}$

Ans.: (d)

Solution: $E_0^1 = \frac{2}{L}\left[\int_0^{\frac{L}{2}} V_0 \sin^2\frac{\pi x}{L}dx + \int_{\frac{L}{2}}^{L} 0\times\sin^2\frac{\pi x}{L}dx\right]$

$$\Rightarrow E_0^1 = \frac{2}{L}\frac{V_0}{2}\int_0^{\frac{L}{2}}\left(1-\cos\frac{2\pi x}{L}\right)dx = \frac{V_0}{L}\left[x - \sin\left(\frac{2\pi x}{L}\right)\frac{L}{2\pi}\right]_0^{\frac{L}{2}}$$

$$\Rightarrow E_0^1 = \frac{V_0}{2} \Rightarrow E = \frac{\hbar^2\pi^2}{2mL^2} + \frac{V_0}{2}$$

Q40. Let the Hamiltonian for two spin-½ particles of equal masses m, momenta $\vec{p}_1$ and $\vec{p}_2$ and positions $\vec{r}_1$ and $\vec{r}_2$ be $H=\frac{1}{2m}p_1^2+\frac{1}{2m}p_2^2+\frac{1}{2}m\omega^2\left(r_1^2+r_2^2\right)+k\vec{\sigma}_1\cdot\vec{\sigma}_2$, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ denote the corresponding Pauli matrices, $\hbar\omega=0.1eV$ and $k=0.2\ eV$. If the ground state has net spin zero, then the energy (in eV) is _____

Ans.: –0.3

Solution: $H=\frac{1}{2m}p_1^{\ 2}+\frac{1}{2m}p_2^2+\frac{1}{2}m\omega^2\left(r_1^2+r_2^2\right)+k\vec{\sigma_1}.\vec{\sigma_2}$

$\vec{\sigma}=\vec{\sigma_1}+\vec{\sigma}_2 \quad\Rightarrow\vec{\sigma}^2=\sigma_1^2+\sigma_2^2+2\vec{\sigma_1}.\vec{\sigma_2}\Rightarrow 2\vec{\sigma_1}.\vec{\sigma_2}=\vec{\sigma}^2-\sigma_1^2-\sigma_2^2$

$\Rightarrow 2\vec{\sigma_1}.\vec{\sigma_2}=0-3I-3I=-6I\Rightarrow\vec{\sigma_1}.\vec{\sigma_2}=-3$

Now energy $E=2\times\frac{3}{2}\hbar\omega+k(-3)=3\times(0.1)+(0.2)(-3)=-0.3\,eV$

Q41. Suppose a linear harmonic oscillator of frequency ω and mass m is in the state $|\psi\rangle=\frac{1}{\sqrt{2}}\left[|\psi_0\rangle+e^{i\frac{\pi}{2}}|\psi_1\rangle\right]$ at $t=0$ where $|\psi_0\rangle$ and $|\psi_1\rangle$ are the ground and the first excited states, respectively. The value of $\langle\psi|x|\psi\rangle$ in the units of $\sqrt{\frac{\hbar}{m\omega}}$ at $t=0$ is _____

Ans. : 0

Solution: $|\psi\rangle=\frac{1}{\sqrt{2}}\left[|\psi_0\rangle+e^{\frac{i\pi}{2}}|\psi_1\rangle\right]$

$\langle\psi|x|\psi\rangle=\langle\psi|\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)|\psi\rangle\Rightarrow\langle\psi|x|\psi\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left(\langle\psi|a|\psi\rangle+\langle\psi|a^\dagger|\psi\rangle\right)$

$\because a|\psi\rangle=\frac{1}{\sqrt{2}}e^{\frac{i\pi}{2}}|\psi_0\rangle \quad\text{and}\quad a^\dagger|\psi\rangle=\frac{1}{\sqrt{2}}\left(|\psi_1\rangle+\sqrt{2}e^{\frac{i\pi}{2}}|\psi_2\rangle\right)$

$\Rightarrow\langle\psi|x|\psi\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{\frac{i\pi}{2}}\langle\psi_0|\psi_0\rangle+\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{-\frac{i\pi}{2}}\langle\psi_1|\psi_1\rangle\right)$

$\Rightarrow\langle\psi|x|\psi\rangle=\sqrt{\frac{\hbar}{2m\omega}}(0)=0$

GATE-2016

Q42. Which of the following operators is Hermitian?

(a) $\frac{d}{dx}$ (b) $\frac{d^2}{dx^2}$ (c) $i\frac{d^2}{dx^2}$ (d) $\frac{d^3}{dx^3}$

Ans. : (b)

Q43. The scattering of particles by a potential can be analyzed by Born approximation. In particular, if the scattered wave is replaced by an appropriate plane wave, the corresponding Born approximation is known as the first Born approximation. Such an approximation is valid for

(a) large incident energies and weak scattering potentials.

(b) large incident energies and strong scattering potentials.

(c) small incident energies and weak scattering potentials.

(d) small incident energies and strong scattering potentials.

Ans.: (a)

Q44. Consider an elastic scattering of particles in $l=0$ states. If the corresponding phase shift δ_0 is 90^0 and the magnitude of the incident wave vector is equal to $\sqrt{2\pi}\ fm^{-1}$ then the total scattering cross section in units of fm^2 is _____.

Ans.: 2

Solution: $\sigma=\frac{4\pi}{k^2}\sum_{l=0}(2l+1)\sin\delta_0$ for $l=0$, it is given $\delta_0=90^0$ and $k=\sqrt{2\pi}\ fm^{-1}$

$$\sigma=\frac{4\pi}{2\pi}\sin 90=2$$

Q45. A hydrogen atom is in its ground state. In the presence of a uniform electric field $\vec{E}=E_0\hat{z}$, the leading order change in its energy is proportional to $(E_0)^n$. The value of the exponent n is _____.

Ans.: 2

Solution: First order energy correction is zero $\langle\psi_{1,0,0}|E_0 r\cos\theta|\psi_{1,0,0}\rangle=0$

So one need to find correction of second $\sum_{n\neq 1}\frac{\left|\langle\psi_{n,l,m}|E_0 r\cos\theta|\psi_{1,0,0}\rangle\right|^2}{E_1^0-E_m^0}\propto E_0^2$

So value of $n=2$

Q46. If $\vec{s}_1$ and $\vec{s}_2$ are the spin operators of the two electrons of a He atom, the value of $\langle \vec{s}_1.\vec{s}_2 \rangle$ for the ground state is

(a) $-\frac{3}{2}\hbar^2$ (b) $-\frac{3}{4}\hbar^2$ (c) 0 (d) $\frac{1}{4}\hbar^2$

Ans.: (b)

Solution: $\vec{s} = \vec{s}_1 + \vec{s}_2,\ s_1 = \frac{1}{2}, s_1 = \frac{1}{2}, s = 0,1,\ \langle \vec{s}_1.\vec{s}_2 \rangle = \frac{s(s+1)\hbar^2 - s_1(s_1+1)\hbar^2 - s_2(s_2+1)\hbar^2}{2}$

For $s = 1, \langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2}{2} = \frac{3}{4}\hbar^2$

$s = 0, \langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{0\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2}{2} = -\frac{3}{4}\hbar^2$

Q47. A two-dimensional square rigid box of side L contains six non-interacting electrons at $T = 0K$. The mass of the electron is m. The ground state energy of the system of electrons, in units of $\frac{\pi^2\hbar^2}{2mL^2}$ is _____.

Ans.: 24

Solution: $2 \times \frac{(1^2+1^2)\pi^2\hbar^2}{2mL^2} + 4 \times \frac{(2^2+1^2)\pi^2\hbar^2}{2mL^2} = \frac{24\pi^2\hbar^2}{2mL^2}$

Q48. σ_x, σ_y and σ_z are the Pauli matrices. The expression $2\sigma_x\sigma_y + \sigma_y\sigma_x$ is equal to

(a) $-3i\sigma_z$ (b) $-i\sigma_z$ (c) $i\sigma_z$ (d) $3i\sigma_z$

Ans.: (c)

Solution: $2\sigma_x\sigma_y + \sigma_y\sigma_x \Rightarrow \sigma_x\sigma_y + \sigma_x\sigma_y + \sigma_y\sigma_x \Rightarrow \sigma_x\sigma_y = i\sigma_z$

Q49. If x and p are the x components of the position and the momentum operators of a particle respectively, the commutator $[x^2, p^2]$ is

(a) $i\hbar(xp - px)$ (b) $2i\hbar(xp - px)$ (c) $i\hbar(xp + px)$ (d) $2i\hbar(xp + px)$

Ans.: (d)

Solution: $[x^2, p^2] = p[x^2, p] + [x^2 p]p = 2i\hbar px + 2i\hbar xp \Rightarrow 2i\hbar(xp + px)$

Q50. Let $|l,m\rangle$ be the simultaneous eigenstates of L^2 and L_z. Here $\vec{L}$ is the angular momentum operator with Cartesian components (L_x, L_y, L_z), l is the angular momentum quantum number and m is the azimuthal quantum number. The value of $\langle 1,0|(L_x + iL_y)|1,-1\rangle$ is

(a) 0 (b) $\hbar$ (c) $\sqrt{2}\hbar$ (d) $\sqrt{3}\hbar$

Ans.: (c)

Solution: $\langle 1,0|(L_x + iL_y|1,-1\rangle = \langle 1,0|L_+|1,-1\rangle = \sqrt{2}\hbar\langle 1,0|1,0\rangle = \sqrt{2}\hbar$

Q51. For the parity operator P, which of the following statements is **NOT** true?

(a) $P^\dagger = P$ (b) $P^2 = -P$ (c) $P^2 = I$ (d) $P^\dagger = P^{-1}$

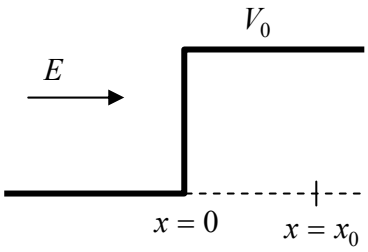
Ans.: (b)

Q52. The state of a system is given by $|\psi\rangle = |\phi_1\rangle + 2|\phi_2\rangle + 3|\phi_3\rangle$, where $|\phi_1\rangle, |\phi_2\rangle and |\phi_3\rangle$ form an orthonormal set. The probability of finding the system in the state $|\phi_2\rangle$ is _____. (Give your answer upto two decimal places)

Ans. : 0.28

Solution: Probability that ψ in state $|\phi_2\rangle = \frac{2^2}{1^2+2^2+3^2} = \frac{4}{1+4+9} = \frac{4}{14} = \frac{2}{7} = 0.28$

Q53. A particle of mass m and energy E, moving in the positive x direction, is incident on a step potential at $x = 0$, as indicated in the figure. The height of the potential is V_0, where $V_0 > E$. At $x = x_0$, where $x_0 > 0$, the probability of finding the electron is $\frac{1}{e}$ times the probability of finding it at $x = 0$. If $\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$, the value of x_0 is



(a) $\frac{2}{\alpha}$ (b) $\frac{1}{\alpha}$ (c) $\frac{1}{2\alpha}$ (d) $\frac{1}{4\alpha}$

Ans.: (c)

Solution: $\frac{1}{e} = e^{-2\alpha x_0} = e^{-1} = e^{-2\alpha x_0} \Rightarrow x_0 = \frac{1}{2\alpha}$

GATE- 2017

Q54. The Compton wavelength of a proton is…….. fm. (up to two decimal places).

Ans. : 3×10^{8}

Solution: $\left(m_p = 1.67\times10^{-27}\,kg, h = 6.626\times10^{-34}\,Js, e = 1.602\times10^{-19}\,C, c = 3\times10^{8}\,ms^{-1}\right)$

Q55. A one dimensional simple harmonic oscillator with Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2$ is subjected to a small perturbation, $H_1 = \alpha x + \beta x^3 + \gamma x^4$. The first order correction to the ground state energy is dependent on

(a) only β (b) α and γ (c) α and β (d) only γ

Ans. : (d)

Solution: $H_1 = \alpha x + \beta x^3 + \gamma x^4$, $E_g^1 = \alpha\langle x\rangle + \beta\langle x^3\rangle + \gamma\langle x^4\rangle$, $\langle x\rangle = 0, \langle x^3\rangle = 0, \langle x^4\rangle \neq 0$

Q56. For the Hamiltonian $H = a_0 I + \vec{b}.\vec{\sigma}$ where $a_0 \in R, \vec{b}$ is a real vector, I is the 2×2 identity matrix, and $\vec{\sigma}$ are the Pauli matrices, the ground state energy is

(a) $|b|$ (b) $2a_0 - |b|$ (c) $a_0 - |b|$ (d) a_0

Ans. : (c)

Solution: $a_0 I + \vec{b}.\vec{\sigma} = a_0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + b_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + b_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + b_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}a_0 + b_z & b_x - ib_y\\ b_x + ib_y & a_0 - b_z\end{pmatrix}$

$$H = a_0 I + \vec{b}.\vec{\sigma} = \begin{pmatrix}a_0 + b_z & b_x - ib_y\\ b_x + ib_y & a_0 - b_z\end{pmatrix}$$

For eigen value $\begin{pmatrix}a_0 + b_z - \lambda & b_x - ib_y\\ b_x + ib_y & a_0 - b_z - \lambda\end{pmatrix} = 0$

$$(a_0 + b_z - \lambda)(a_0 - b_z - \lambda) - (b_x^2 + b_y^2) = 0$$

$$\lambda_1 = a_0 - |b|, \lambda_1 = a_0 + |b|$$

Q57. The degeneracy of the third energy level of a 3-dimensional isotropic quantum harmonic oscillator is

(a) 6 (b) 12 (c) 8 (d) 10

Ans. : (a)

Solution: First energy level is $n=0$

Second energy level is $n=1$

Third energy level is $n=2$

Degeneracy of third level $\frac{(n+1)(n+2)}{2}=\frac{3\times 4}{2}=6$

Q58. A free electron of energy $1eV$ is incident upon a one-dimensional finite potential step of height $0.75eV$. The probability of its reflection from the barrier is……. (up to two decimal places).

Ans. : 0.11

Solution: $R=\left(\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2=\left(\frac{1-\sqrt{0.25}}{1+\sqrt{0.25}}\right)^2=\left(\frac{1-0.5}{1+0.5}\right)^2=0.11$

Q59. Consider a one-dimensional potential well of width $3nm$. Using the uncertainty principle $\left(\Delta x\cdot\Delta p\geq\frac{\hbar}{2}\right)$, an estimate of the minimum depth of the well such that it has at least one bound state for an electron is ($m_e=9.31\times10^{-31}kg$, $h=6.626\times10^{-34}Js$, $e=1.602\times10^{-19}C$)

(a) $1\mu eV$ (b) $1meV$ (c) $1eV$ (d) $1MeV$

Ans. : (b)

Solution: $E=\frac{p^2}{2m}$, $\Delta p=\frac{\hbar}{2\Delta x}\Rightarrow\Delta p=\frac{\hbar}{2a}$

So, $E=\frac{\hbar^2}{8ma^2}=\frac{h^2}{32\pi^2ma^2}=\frac{\left(6.6\times10^{-34}\right)^2}{32\times10\times9.31\times10^{-31}\times9\times10^{-18}}=.001\times10^{-19}J\approx1meV$

Q60. The integral $\int_0^\infty x^2e^{-x^2}dx$ is equal to…….. (up to two decimal places).

Ans. : 0.44

Solution: The given integral is $\int_0^\infty x^2\,e^{-x^2}\,dx$

Let $x^2=t$ then $2xdx=dt\Rightarrow dx=\frac{dt}{2\sqrt{t}}$

Thus, the given integral can be written as

$$\int_0^\infty t\,e^{-t}\,\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^\infty e^{-t}\,t^{1/2}\,dt=\frac{1}{2}\int_0^\infty e^{-t}\,t^{\frac{3}{2}-1}\,dt=\frac{1}{2}\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{4}$$

Hence the value of the integral up to two decimal places is 0.44.

Q61. Which one of the following operators is Hermitian?

(a) $i\dfrac{\left(p_x x^2-x^2 p_x\right)}{2}$ (b) $i\dfrac{\left(p_x x^2+x^2 p_x\right)}{2}$

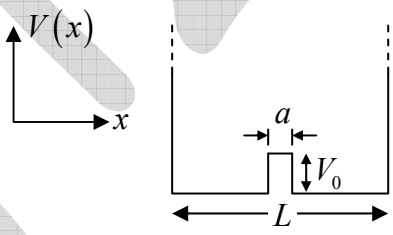
(c) $e^{i p_x a}$ (d) $e^{-i p_x a}$

Ans. : (a)

Solution: $A=i\dfrac{\left(p_x x^2-x^2 p_x\right)}{2}, A^\dagger=-i\dfrac{\left(\left(p_x x^2\right)^\dagger-\left(x^2 p_x\right)^\dagger\right)}{2}=i\dfrac{\left(p_x x^2-x^2 p_x\right)}{2}$

GATE-2018

Q62. The ground state energy of a particle of mass m in an infinite potential well is E_0. It changes to $E_0\left(1+\alpha\times10^{-3}\right)$, when there is a small potential pump of height $V_0=\dfrac{\pi^2\hbar^2}{50\,mL^2}$ and width $a=L/100$, as shown in the figure. The value of α is _____ (up to two decimal places).



Ans. : 0.81

Solution: $\alpha_1=\left(\dfrac{L}{2}-\dfrac{a}{2}\right), \alpha_2=\left(\dfrac{L}{2}+\dfrac{a}{2}\right), \quad a=\dfrac{L}{100}$

$$E_1=V_0\int_{\alpha_1}^{\alpha_2}\left(\sqrt{\frac{2}{L}}\right)^2\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$=\frac{V_0}{L}\int_{\alpha_1}^{\alpha_2}\left[1-\cos\frac{2\pi x}{L}\right]dx=\frac{V_0}{L}\left[x-\frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_{\alpha_1}^{\alpha_2}$$

$$= \frac{V_0}{L}\left[a - \frac{L}{2\pi}\left(\sin\frac{2\pi(L+a)}{2L} - \sin\frac{2\pi(L-a)}{2L}\right)\right]$$

$$= \frac{V_0}{L}\left[\frac{L}{100} - \frac{L}{2\pi}\left(\sin\left(\pi + \frac{\pi a}{L}\right) - \sin\left(\pi - \frac{\pi a}{L}\right)\right)\right]$$

$$= V_0\left[\frac{1}{100} + \frac{1}{2\pi}(0.0314 + 0.0314)\right]$$

$$= V_0 \times 10^{-3}(10+10) = E_0 \times 10^{-3} \times \left(\frac{20}{25}\right) \Rightarrow \alpha E_0 \times 10^{-3} = 0.81 \times E_0 \times 10^{-3}$$

Hence, $\alpha = 0.81$

Q63. A two-state quantum system has energy eigenvalues $\pm \in$ corresponding to the normalized states $|\psi_\pm\rangle$. At time $t = 0$, the system is in quantum state $\frac{1}{\sqrt{2}}\left[|\psi_+\rangle + |\psi_-\rangle\right]$. The probability that the system will be in the same state at $t = h/(6\in)$ is _____ (up to two decimal places).

Ans. : 0.25

Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|\psi_+\rangle + |\psi_-\rangle\right]$

And $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|\psi_+\rangle e^{-\frac{i\in t}{\hbar}} + |\psi_-\rangle e^{\frac{i\in t}{\hbar}}\right]$

At $t = \frac{\hbar}{6\in}$,

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|\psi_+\rangle e^{-\frac{i\in h\times 2\pi}{6\in h}} + |\psi\rangle e^{\frac{i\in h\times 2\pi}{6\in h}}\right] = \frac{1}{\sqrt{2}}\left[|\psi_+\rangle e^{\frac{-i\pi}{3}} + |\psi_-\rangle e^{\frac{i\pi}{3}}\right]$$

Now, probability in same state

$$P = \frac{\left|\langle\psi(t)|\psi(0)\rangle\right|^2}{\langle\psi|\psi\rangle} = \frac{1}{4}\left|e^{-i\pi/3} + e^{i\pi/3}\right|^2 = \frac{1}{4}\left|2\cos\frac{\pi}{3}\right|^2 = \frac{1}{4}\times\left|2\times\frac{1}{2}\right|^2 = 0.25$$

GATE-2019

Q64. An electric field $\vec{E} = E_0\hat{z}$ is applied to a Hydrogen atom in $n = 2$ excited state. Ignoring spin the $n = 2$ state is fourfold degenerate, which in the $|l,m\rangle$ basis are given by $|0,0\rangle, |1,1\rangle, |1,0\rangle$ and $|1,-1\rangle$. If H' is the interaction Hamiltonian corresponding to the applied electric field, which of the following matrix elements is nonzero?

(a) $\langle 0,0|H'|0,0\rangle$ (b) $\langle 0,0|H'|1,1\rangle$

(c) $\langle 0,0|H'|1,0\rangle$ (d) $\langle 0,0|H'|1,-1\rangle$

Ans. : (c)

Q65. For a spin $\frac{1}{2}$ particle, let $|\uparrow\rangle$ and $|\downarrow\rangle$ denote its spin up and spin down states respectively. If $|a\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$ and $|b\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$ are composite states of two such particles, which of the following statements is true for their total spin S?

(a) $S = 1$ for $|a\rangle$ and $|b\rangle$ is not an eigenstate of the operator $\hat{S}^2$

(b) $|a\rangle$ is not an eigenstate of the operator $\hat{S}^2$ and $S = 0$ for $|b\rangle$

(c) $S = 0$ for $|a\rangle$, and $S = 1$ for $|b\rangle$

(d) $S = 1$ for $|a\rangle$, and $S = 0$ for $|b\rangle$

Ans. : (d)

Solution: $S = 1$ is triplet $|a\rangle$, and $S = 0$ for singlet for $|b\rangle$

Q66. The Hamiltonian for a quantum harmonic oscillator of mass m in three dimensions is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

where ω is the angular frequency. The expectation value of r^2 in the first excited state of the oscillator in units of $\frac{\hbar}{m\omega}$ (rounded off to one decimal place) is_____

Ans. : 2.5

Solution: $\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle$

$$= \frac{\hbar}{2m\omega}\left[(2n_x+1)+(2n_y+1)+(2n_z+1)\right]$$

For first excited state $n_x = 1, n_y = 0, n_z = 0$

Hence it is triply degenerate one can take

$n_x = 0,\ n_y = 1,\ n_z = 0$ or $n_x = 0, n_y = 0,\ n_z = 1$

putting any one combination, expectation value of $r^2 = \frac{5}{2}\frac{\hbar}{m\omega} = 2.5\frac{\hbar}{m\omega}$

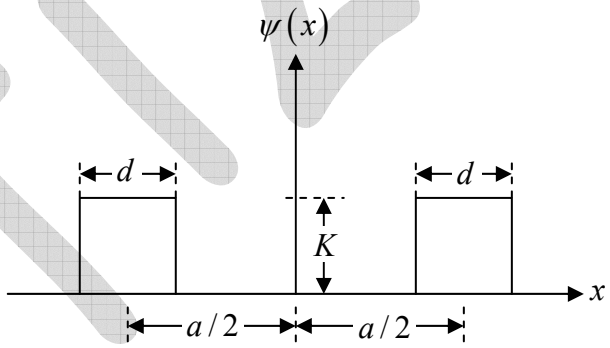
Q67. Let $|\psi_2\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ represent two possible states of a two-level quantum system. The state obtained by the incoherent superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by a density matrix that is defined as $\rho \equiv c_1|\psi_1\rangle\langle\psi_1| + c_2|\psi_2\rangle\langle\psi_2|$. If $c_1 = 0.4$ and $c_2 = 0.6$, the matrix element ρ_{22} (rounded off to one decimal place) is _____

Ans. : 0.6

Solution: $\rho_{2,2} = \langle\psi_2|\rho|\psi_2\rangle = \rho \equiv c_1\langle\psi_2|\psi_1\rangle\langle\psi_1|\psi_2\rangle + c_2\langle\psi_2|\psi_2\rangle\langle\psi_2|\psi_2\rangle$

$\Rightarrow c_2 = 0.6$

Q68. The wave function $\psi(x)$ of a particle is as shown below



Here K is a constant, and $a > d$. The position uncertainty (Δx) of the particle is

(a) $\sqrt{\frac{a^2+3d^2}{12}}$ (b) $\sqrt{\frac{3a^2+d^2}{12}}$ (c) $\sqrt{\frac{d^2}{6}}$ (d) $\sqrt{\frac{d^2}{24}}$

Ans. : (b)

Solution: $\psi(x)=\begin{cases} k, & -\frac{a}{2}-\frac{d}{2}<x<-\frac{a}{2}+\frac{d}{2} \\ 0, & -\frac{a}{2}+\frac{d}{2}<x<\frac{a}{2}-\frac{d}{2} \\ k, & \frac{a}{2}-\frac{d}{2}<x<\frac{a}{2}+\frac{d}{2} \\ 0, & \frac{a}{2}+\frac{d}{2}>0 \end{cases}$

$\langle\psi|\psi\rangle=1$

$$k^2\int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}dx+k^2\int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}dx=1$$

$$k^2\left[\left(-\frac{a}{2}+\frac{d}{2}\right)-\left(-\frac{a}{2}-\frac{d}{2}\right)\right]+k^2\left[\left(\frac{a}{2}+\frac{d}{2}\right)-\left(\frac{a}{2}-\frac{d}{2}\right)\right]=1$$

$$k^2\left[\frac{d}{2}+\frac{d}{2}+\frac{d}{2}+\frac{d}{2}\right]=1\Rightarrow k=\frac{1}{\sqrt{2d}}$$

Hence wavefunction is symmetric about $x=0$, so $\langle x\rangle=0$

$$\langle x^2\rangle=k^2\int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}x^2dx+k^2\int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}x^2dx$$

$$=\frac{k^2}{3}\left[\left[x^3\right]_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}+\left[x^3\right]_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}\right]$$

$$=\frac{k^2}{3\times8}\left[(-a+d)^3-(-a-d)^3+(a+d)^3-(a-d)^3\right]$$

$$=\frac{k^2}{24}\left\{\left(-a^3+d^3-3a^2d+3ad^2\right)+\left(a^3+d^3+3a^2d+3ad^2\right)+\left(a^3+d^3+3a^2d+3ad^2\right)\right.$$
$$\left.-\left\{a^3-d^3-3ad(a-d)\right\}\right]$$

$$\langle x^2\rangle=\frac{k^2}{24}\left[4a^3+12a^2d\right]=\frac{4d\left(d^2+3a^2\right)}{24\times2d}=\frac{3a^2+d^2}{12}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3a^2 + d^2}{12}}$$

Q69. Consider the motion of a particle along the x - axis in a potential $V(x) = F|x|$. Its ground state energy E_0 is estimated using the uncertainty principle. Then E_0 is proportional to

(a) $F^{1/3}$ (b) $F^{1/2}$ (c) $F^{2/5}$ (d) $F^{2/3}$

Ans. : (d)

Solution: $E = \frac{p^2}{2m} + F|x|$ $E = \frac{p^2}{2m} + Fx$ for $x > 0$ $E = \frac{p^2}{2m} - Fx$ < 0 from uncertainty theory

$$\Delta x.\Delta p = \hbar \Rightarrow \Delta p = \frac{\hbar}{\Delta x}$$

$$E = \frac{(\Delta p)^2}{2m} + F(\Delta x) \Rightarrow E = \frac{\hbar^2}{2m(\Delta x)^2} + F\Delta x$$

For minimum energy,

$$\frac{dE}{d\Delta x} = -\frac{\hbar^2}{m(\Delta x)^3} + F = 0 \Rightarrow \Delta x = \left(\frac{\hbar^2}{mF}\right)^{1/3} \quad \frac{\hbar^2}{2m}\left(\frac{mF}{\hbar^2}\right)^{2/3} + F\left(\frac{\hbar^2}{mF}\right)^{1/3} \Rightarrow E \propto F^{2/3}$$

Q70. The Hamiltonian operator for a two-level quantum system is $H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$. If the state of the system at $t = 0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $\left|\langle \psi(0)|\psi(t)\rangle\right|^2$ at a later time t is

(a) $\frac{1}{2}\left(1 + e^{-(E_1 - E_2)t/\hbar}\right)$ (b) $\frac{1}{2}\left(1 - e^{-(E_1 - E_2)t/\hbar}\right)$

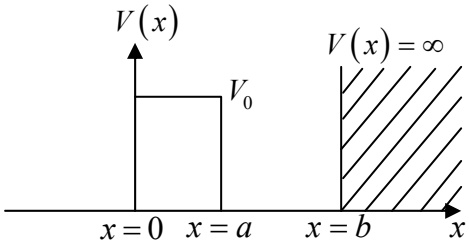
(c) $\frac{1}{2}\left(1 + \cos[E_1 - E_2]t/\hbar\right)$ (d) $\frac{1}{2}\left(1 - \cos[E_1 - E_2]t/\hbar\right)$

Ans. : (c)

Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp -\frac{iE_1 t}{\hbar} \\ \exp -\frac{iE_2 t}{\hbar} \end{pmatrix}$

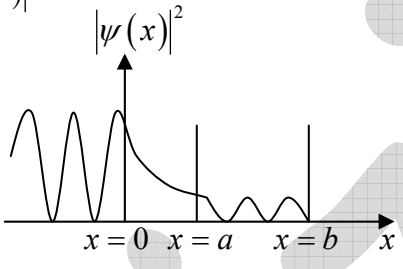
$$\left|\langle\psi(0)|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|\exp-\frac{iE_1t}{\hbar}+\exp-\frac{iE_2t}{\hbar}\right|^2 = \frac{1}{2}\left(1+\cos\left[E_1-E_2\right]t/\hbar\right)$$

Q71. Consider a potential barrier $V(x)$ of the form:

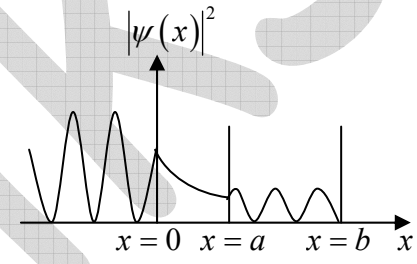


where V_0 is a constant. For particles of energy $E < V_0$ incident on this barrier from the left which of the following schematic diagrams best represents the probability density $|\psi(x)|^2$ as a function of x?

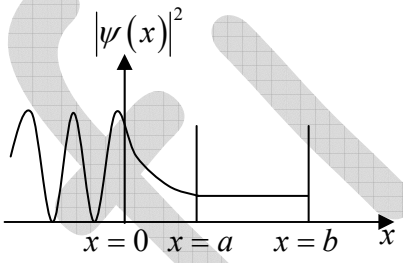
(a)



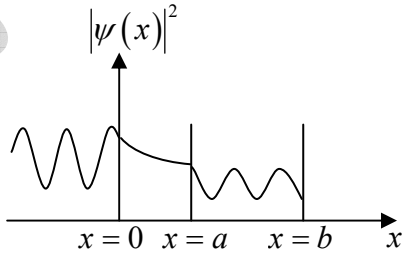
(b)



(c)



(d)



Ans. : (a)

Q72. The Hamiltonian of a system is $H = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & -1 \end{pmatrix}$ with $\varepsilon \ll 1$. The fourth order contribution to the ground state energy of H is $\gamma\varepsilon^4$. The value of γ (rounded off to three decimal places) is_____.

Ans. : 0.125

Solution: $H = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & -1 \end{pmatrix}$ the eigen value of the hamiltonion is $E_g = -\sqrt{1-\varepsilon^2}, E_f = +\sqrt{1-\varepsilon^2}$

The ground state is $E_g = -\sqrt{1-\varepsilon^2}$

Taylor expansion of $-\sqrt{1-\varepsilon^2} = -\left(1-\frac{\varepsilon^2}{2}-\frac{\varepsilon^4}{8}.....\right) = -1+\frac{\varepsilon^2}{2}+\frac{\varepsilon^4}{8}.....$

$\gamma = \frac{1}{8} = 0.125$

Q73. Electrons with spin in the z - direction $(\hat{z})$ are passed through a Stern-Gerlach (SG) set up with the magnetic field at $\theta = 60^0$ from $\hat{z}$. The fraction of electrons that will emerge with their spin parallel to the magnetic field in the SG set up (rounded off to two decimal places) is_____

$$\left[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

Ans. : 0.25

Solution: $|\psi\rangle = \begin{pmatrix} \cos 60^0 \\ \sin 60^0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$ state related to up state is $\left|\frac{1}{2},\frac{1}{2}\right\rangle = |\chi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The fraction of electrons that will emerge with their spin parallel to the magnetic field $\left|\langle\chi|\psi\rangle\right|^2 = \frac{1}{4} = 0.25$

THERMODYNAMICS AND STATISTICAL PHYSICS SOLUTIONS

GATE 2010

Q1. A system of N non-interacting classical point particles is constrained to move on the two-dimensional surface of a sphere. The internal energy of the system is

(a) $\frac{3}{2}Nk_BT$ (b) $\frac{1}{2}Nk_BT$ (c) Nk_BT (d) $\frac{5}{2}Nk_BT$

Ans: (c)

Solution: There are 2 N degree of freedom.

The internal energy of the system is $\frac{Nk_BT}{2}+\frac{Nk_BT}{2}=Nk_BT$

Q2. Which of the following atoms cannot exhibit Bose-Einstein condensation, even in principle?

(a) $^{1}H_{1}$ (b) $^{4}H_{2}$ (c) $^{23}Na_{11}$ (d) $^{30}K_{19}$

Ans: (d)

Solution: For Bose-Einstein condensation:

Number of electron + number of proton + number of neutron = Even

For $^{30}K_{19}$

Number of proton = 19, Number of electron = 19, Number of neutron = 11.

19 + 19 + 11 = 49 this is odd. So it will not exhibit Bose-Einstein condensation.

Q3. For a two-dimensional free electron gas, the electronic density n, and the Fermi energy E_F, are related by

(a) $n=\frac{(2mE_F)^{3/2}}{3\pi^2\hbar^3}$ (b) $n=\frac{mE_F}{\pi\hbar^2}$

(c) $n=\frac{mE_F}{2\pi\hbar^2}$ (d) $n=\frac{(2mE_F)^{3/2}}{\pi\hbar}$

Ans: (c)

Solution: $n=\int_0^{E_F} g(E)f(E)dE$, $g(E)dE=\frac{2m}{h^2}dE$

At $T=0$, $f(E)=\begin{cases}1, & if\ E<E_F\\ 0, & if\ E>E_F\end{cases}\Rightarrow n=\frac{2mE_F}{h^2}=\frac{mE_F}{2\pi^2\hbar^2}$

Q4. Which among the following sets of Maxwell relations is correct? (U-internal energy, H-enthalpy, A-Helmholtz free energy and G-Gibbs free energy)

(a) $T=\left(\frac{\partial U}{\partial V}\right)_S$ and $P=\left(\frac{\partial U}{\partial S}\right)_V$

(b) $V=\left(\frac{\partial H}{\partial P}\right)_S$ and $T=\left(\frac{\partial H}{\partial S}\right)_P$

(c) $P=-\left(\frac{\partial G}{\partial V}\right)_T$ and $V=\left(\frac{\partial G}{\partial P}\right)_S$

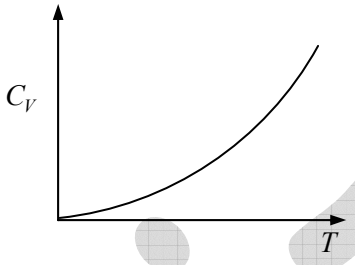
(d) $P=-\left(\frac{\partial A}{\partial S}\right)_T$ and $S=\left(\frac{\partial A}{\partial P}\right)_V$

Ans: (b)

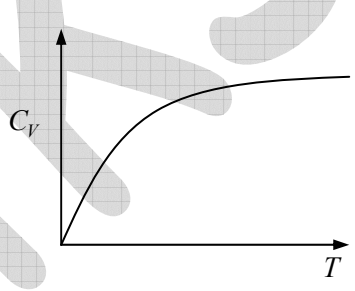
Solution: $dH=TdS+VdP \Rightarrow \left(\frac{\partial H}{\partial S}\right)_P=T, \left(\frac{\partial H}{\partial P}\right)_S=V$

Q5. Partition function for a gas of photons is given as, $\ln Z=\frac{\pi^2 V\left(k_B T\right)^3}{45\hbar^3 C^3}$. The specific heat of the photon gas varies with temperature as

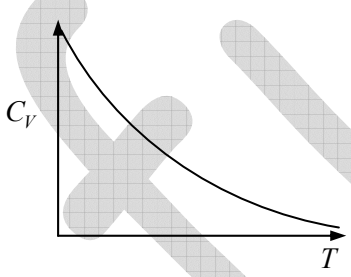
(a)



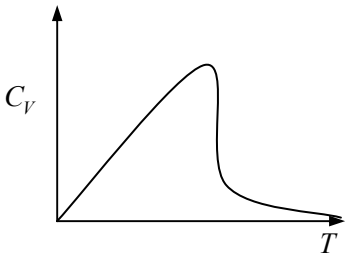
(b)



(c)



(d)



Ans: (a)

Solution: $U=K_B T^2 \frac{\partial \ln z}{\partial T}$, $C_V=\left(\frac{\partial U}{\partial T}\right)_v \Rightarrow C_V \propto T^3$.

Q6. From Q. no. 5, the pressure of the photon gas is

(a) $\frac{\pi^2\left(k_B T\right)^3}{15\hbar^3 C^3}$ (b) $\frac{\pi^2\left(k_B T\right)^4}{8\hbar^3 C^3}$ (c) $\frac{\pi^2\left(k_B T\right)^4}{45\hbar^3 C^3}$ (d) $\frac{\pi^2\left(k_B T\right)^{3/2}}{45\hbar^3 C^3}$

Ans: (c)

Solution: Since, $P=-\frac{\partial F}{\partial V} \Rightarrow P=KT\left(\frac{\partial \ln z}{\partial V}\right)_T=\frac{\pi^2\left(k_0 T\right)^4}{45\hbar^3 C^3}$

GATE 2011

Q7. A Carnot cycle operates on a working substance between two reservoir at temperatures T_1 and T_2 with $T_1 > T_2$. During each cycle, an amount of heat Q_1 is extracted from the reservoir at T_1 and an amount Q_2 is delivered in the reservoir at T_2. Which of the following statements is **INCORRECT**?

(a) Work done in one cycle is $Q_1 - Q_2$

(b) $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

(c) Entropy of the hotter reservoir decreases

(d) Entropy of the universe (consisting of the working substance and the two reservoirs) increases

Ans: (c)

Solution: Entropy of hotter reservoirs decreases.

Q8. In a first order phase transition, at the transition temperature, specific heat of the system

(a) diverges and its entropy remains the same

(b) diverges and its entropy has finite discontinuity

(c) remains unchanged and its entropy has finite discontinuity

(d) has finite discontinuity and its entropy diverges

Ans: (b)

Q9. A system of N non-interacting and distinguishable particle of spin 1 is in thermodynamic equilibrium. The entropy of the system is

(a) $2k_B \ln N$ (b) $3k_B \ln N$ (c) $Nk_B \ln 2$ (d) $Nk_B \ln 3$

Ans: (d)

Solution: $S = k_B \sum_i \ln \Omega$, $\Omega = 3$ is number of microstate. $S = 1$; $S_z = -1, \ 0, \ 1$

The entropy of the system is $Nk_B \ln 3$.

Q10. A system has two energy levels with energies ε and 2ε. The lower level is 4-fold degenerate while the upper level is doubly degenerate. If there are N non-interacting classical particles in the system, which is in thermodynamic equilibrium at a temperature T, the fraction of particles in the upper level is

(a) $\frac{1}{1+e^{\varepsilon/k_BT}}$ (b) $\frac{1}{1+2e^{\varepsilon/k_BT}}$

(c) $\frac{1}{2e^{\varepsilon/k_BT}+4e^{2\varepsilon/k_BT}}$ (d) $\frac{1}{2e^{\varepsilon/k_BT}-4e^{2\varepsilon/k_BT}}$

Ans: (b)

Solution: Partition function $Z=4e^{-\in/kT}+2e^{-\in/kT}\Rightarrow P(2\varepsilon)=\frac{2e^{-2\in/kT}}{4e^{-\in/kT}+2e^{-2\in/kT}}=\frac{1}{1+2e^{\in/kT}}$

GATE 2012

Q11. The isothermal compressibility, κ of an ideal gas at temperatures T_0 and V_0 is given by

(a) $-\frac{1}{V_0}\frac{\partial V}{\partial P}\Big|_{T_0}$ (b) $\frac{1}{V_0}\frac{\partial V}{\partial P}\Big|_{T_0}$ (c) $-V_0\frac{\partial P}{\partial V}\Big|_{T_0}$ (d) $V_0\frac{\partial P}{\partial V}\Big|_{T_0}$

Ans: (c)

Solution: Isothermal compressibility $\kappa=-\mathrm{V}\left(\frac{\partial \mathrm{P}}{\partial \mathrm{V}}\right)_{\mathrm{T}}$

Q12. For an ideal Fermi gas in three dimensions, the electron velocity V_F at the Fermi surface is related to electron concentration n as,

(a) $V_F\propto n^{2/3}$ (b) $V_F\propto n$ (c) $V_F\propto n^{1/2}$ (d) $V_F\propto n^{1/3}$

Ans: (d)

Solution: $E_F=\frac{1}{2}mV_F^2\quad\because E_F\propto n^{2/3}\Rightarrow V_F^2\propto n^{2/3}\Rightarrow V_F\propto n^{1/3}$.

Q13. A classical gas of molecules, each of mass m, is in thermal equilibrium at the absolute temperature T. The velocity components of the molecules along the Cartesian axes are v_x, v_y and v_z. The mean value of $(v_x+v_y)^2$ is

(a) $\frac{k_BT}{m}$ (b) $\frac{3}{2}\frac{k_BT}{m}$ (c) $\frac{1}{2}\frac{k_BT}{m}$ (d) $\frac{2k_BT}{m}$

Ans: (d)

Solution: $\left\langle (V_x + V_y)^2 \right\rangle = \left\langle v_x^2 \right\rangle + \left\langle v_y^2 \right\rangle + 2\left\langle v_x \cdot v_y \right\rangle = \left\langle v_x^2 \right\rangle + \left\langle v_y^2 \right\rangle + 2\left\langle v_x \right\rangle \cdot \left\langle v_y \right\rangle = \frac{2k_B T}{m}$

$\because \left\langle v_x \right\rangle = \left\langle v_y \right\rangle = 0$ and $\left\langle V_x^2 \right\rangle + \left\langle V_y^2 \right\rangle = \frac{2k_B T}{m}$.

Q14. The total energy, E of an ideal non-relativistic Fermi gas in three dimensions is given by $E \propto \frac{N^{5/3}}{V^{2/3}}$, where N is the number of particles and V is the volume of the gas. Identify the **CORRECT** equation of state (P being the pressure),

(a) $PV = \frac{1}{3}E$ (b) $PV = \frac{2}{3}E$ (c) $PV = E$ (d) $PV = \frac{5}{3}E$

Ans: (b)

Solution: $P = -\left(\frac{\partial E}{\partial V}\right)_N = \frac{2}{3}\left(\frac{N}{V}\right)^{\frac{5}{3}} \Rightarrow PV = \frac{2}{3}\frac{N^{5/3}}{V^{2/3}} = \frac{2}{3}E$.

Q15. Consider a system whose three energy levels are given by 0, ε and 2ε. The energy level ε is two-fold degenerate and the other two are non-degenerate. The partition function of the system with $\beta = \frac{1}{k_B T}$ is given by

(a) $1 + 2e^{-\beta\varepsilon}$ (b) $2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}$ (c) $(1 + e^{-\beta\varepsilon})^2$ (d) $1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}$

Ans: (c)

Solution: $E_1 = 0, E_2 = \varepsilon, E_3 = 2\varepsilon$; $g_1 = 1,\ g_2 = 2, g_3 = 1$ where g_1, g_2 and g_3 are degeneracy.

The partition function $Z = g_1 e^{-\beta \cdot E_1} + g_2 e^{-\beta \cdot E_2} + g_3 e^{-\beta \cdot E_3} = 1 + 2e^{-\beta\varepsilon} + e^{-\beta 2\varepsilon} = \left(1 + e^{-\beta\varepsilon}\right)^2$

GATE 2013

Q16. If Planck's constant were zero, then the total energy contained in a box filled with radiation of all frequencies at temperature T would be (k is the Boltzmann constant and T is nonzero)

(a) zero (b) Infinite (c) $\frac{3}{2}kT$ (d) kT

Ans: (d)

Solution: If Planck's constant were zero, then the system behaved as a classical system and the energy is kT.

Q17. Across a first order phase transition, the free energy is

(a) proportional to the temperature

(b) a discontinuous function of the temperature

(c) a continuous function of the temperature but its first derivative is discontinuous

(d) such that the first derivative with respect to temperature is continuous

Ans: (c)

Q18. Two gases separated by an impermeable but movable partition are allowed to freely exchange energy. At equilibrium, the two sides will have the same

(a) pressure and temperature (b) volume and temperature

(c) pressure and volume (d) volume and energy

Ans: (a)

Q19. The entropy function of a system is given by $S(E) = aE(E_0 - E)$ where a and E_0 are positive constants. The temperature of the system is

(a) negative for some energies (b) increases monotonically with energy

(c) decreases monotonically with energy (d) Zero

Ans: (a)

Solution: From first and second law of thermodynamics

$$TdS = dU - PdV \Rightarrow dS = \frac{1}{T}(dU - PdV) \Rightarrow \left(\frac{\partial S}{\partial E}\right)_V = \frac{1}{T} \quad \because E = U$$

$$S(E) = aE(E_0 - E) \Rightarrow \left(\frac{\partial S}{\partial E}\right)_V = a(E_0 - E) - aE = a(E_0 - 2E) \Rightarrow T = \frac{1}{a(E_0 - 2E)}.$$

Q20. Consider a linear collection of N independent spin ½ particles, each at a fixed location. The entropy of this system is (k is the Boltzmann constant)

(a) zero (b) Nk (c) $\frac{1}{2}Nk$ (d) $Nk\ln(2)$

Ans: (d)

Solution: There are two microstates possible for spin $\frac{1}{2}$ particle, so entropy is given by $Nk\ln(2)$.

Q21. Consider a gas of atoms obeying Maxwell-Boltzmann statistics. The average value of $e^{\vec{a}\cdot\vec{p}}$ over all the moments $\vec{p}$ of each of the particles (where $\vec{a}$ is a constant vector and a is the magnitude, m is the mass of each atom, T is temperature and k is Boltzmann's constant) is,

(a) one (b) zero (c) $e^{-\frac{1}{2}a^2mkT}$ (d) $e^{-\frac{3}{2}a^2mkT}$

Ans: (c)

Solution: $\langle e^{\vec{p}.\vec{a}}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(p_x,p_y,p_z\right)e^{\vec{p}.\vec{a}}dp_xdp_ydp_z$ where $f\left(p_x,p_y,p_z\right)$ is Maxwell probability distribution at temperature T.

$$\langle e^{\vec{p}.\vec{a}}\rangle = \int_{-\infty}^{\infty} A_x e^{-\frac{p_x^2}{2mkT}} e^{p_x a_x}dp_x \int_{-\infty}^{\infty} A_y e^{-\frac{p_y^2}{2mkT}} e^{p_y a_y}dp_y \int_{-\infty}^{\infty} A_z e^{-\frac{p_z^2}{2mkT}} e^{p_z a_z}dp_z$$

$$\langle e^{\vec{p}.\vec{a}}\rangle = e^{\frac{-(a_x^2+a_y^2+a_z^2)mkT}{2}} \int_{-\infty}^{\infty} A_x e^{-\frac{(p_x-mkTa_x)^2}{2mkT}}dp_x \int_{-\infty}^{\infty} A_y e^{-\frac{(p_y-mkTa_y)^2}{2mkT}}dp_y \int_{-\infty}^{\infty} A_z e^{-\frac{(p_z-mkTa_z)^2}{2mkT}}dp_z$$

$$\langle e^{\vec{p}.\vec{a}}\rangle = e^{\frac{-(a_x^2+a_y^2+a_z^2)mkT}{2}}.1.1.1 = e^{-\frac{1}{2}a^2mkT}$$

Common Data for Questions 22 and 23: There are four energy levels E, $2E, 3E$ and $4E$ (where $E>0$). The canonical partition function of two particles is, if these particles are

Q22. Two identical fermions

(a) $e^{-2\beta E}+e^{-4\beta E}+e^{-6\beta E}+e^{-8\beta E}$ (b) $e^{-3\beta E}+e^{-4\beta E}+e^{-5\beta E}+e^{-6\beta E}+e^{-7\beta E}$

(c) $\left(e^{-\beta E}+e^{-2\beta E}+e^{-3\beta E}+e^{-4\beta E}\right)^2$ (d) $e^{-2\beta E}-e^{-4\beta E}+e^{-6\beta E}-e^{-8\beta E}$

Ans: (b)

Solution: The possible value of Energy for two Fermions

$$E_1=3E, E_2=4E, E_3=5E, E_4=6E, E_5=7E$$

The partition function is $Z=e^{-3\beta E}+e^{-4\beta E}+2e^{-5\beta E}+e^{-6\beta E}+e^{-7\beta E}$, then the answer may be option (b).

Q23. Two distinguishable particles

(a) $e^{-2\beta E}+e^{-4\beta E}+e^{-6\beta E}+e^{-8\beta E}$ (b) $e^{-3\beta E}+e^{-4\beta E}+e^{-5\beta E}+e^{-6\beta E}+e^{-7\beta E}$

(c) $\left(e^{-\beta E}+e^{-2\beta E}+e^{-3\beta E}+e^{-4\beta E}\right)^2$ (d) $e^{-2\beta E}-e^{-4\beta E}+e^{-6\beta E}-e^{-8\beta E}$

Ans: (c)

Solution: When two particles are distinguishable then minimum value of Energy is $2E$ and maximum value is $8E$.

So from checking all four options $\left(Z = e^{-\beta E} + e^{-2\beta E} + e^{-3\beta E} + e^{-4\beta E}\right)^2$

GATE 2014

Q24. For a gas under isothermal condition its pressure P varies with volume V as $P \propto V^{-5/3}$. The bulk modules B is proportional to

(a) $V^{-1/2}$ (b) $V^{-2/3}$ (c) $V^{-3/5}$ (d) $V^{-5/3}$

Ans: (d)

Solution: $P = KV^{-5/3}$, $B = -V\dfrac{dP}{dV}$ $B \propto V^{-5/3}$

Q25. At a given temperature T, the average energy per particle of a non-interacting gas of two-dimensional classical harmonic oscillators is _____ $k_B T$

(k_B is the Boltzmann constant)

Ans: 2

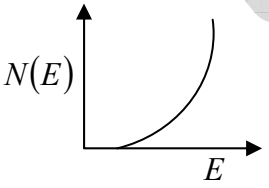
Q26. Which one of the following is a fermion?

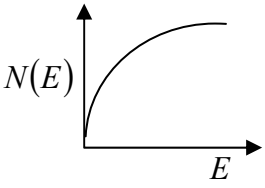
(a) α particle (b) ${}_4Be^2$ nucleus (c) Hydrogen atom (d) deuteron

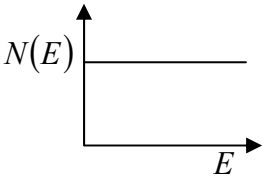
Ans (d)

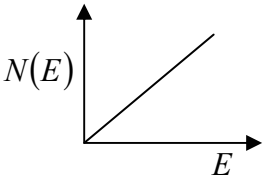
Solution: If total number of particles i.e., electron, proton and neutron is odd, then it is a fermions: $P + N + E = 3$

Q27. For a free electron gas in two dimensions the variations of the density of states. $N(E)$ as a function of energy E, is best represented by

(a) 

(b) 

(c) 

(d) 

Ans. : (c)

Solution: $N(E) \propto E^0$

Q28. For a system of two bosons each of which can occupy any of the two energy levels 0 and ε. The mean energy of the system at temperature T with $\beta = \frac{1}{k_\beta T}$ is given by

(a) $\frac{\varepsilon e^{-\beta\varepsilon} + 2\varepsilon e^{-2\beta\varepsilon}}{1 + 2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$ (b) $\frac{1 + \varepsilon e^{-\beta\varepsilon}}{2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$ (c) $\frac{2\varepsilon e^{-\beta\varepsilon} + \varepsilon e^{-2\beta\varepsilon}}{2 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$ (d) $\frac{\varepsilon e^{-\beta\varepsilon} + 2\varepsilon e^{-2\beta\varepsilon}}{2 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$

Ans. : None of the options are matched.

Solution: If both particle will in ground state the energy will 0, which is non-degenerate. If one particle is in ground state and other is in first excited state then energy is ε and non degenerate. If both particles will in first excited state, then energy will 2ε, which is non-degenerate.

Then partition function is $Z = 1 + \exp(-\beta\varepsilon) + \exp(-2\beta\varepsilon)$

Average value of energy $= \frac{\exp-\beta\varepsilon + 2\varepsilon \exp-2\beta\varepsilon}{1 + \exp-\beta\varepsilon + \exp-2\beta\varepsilon}$

No one answer is correct, but answer may be (a).

Q29. Consider a system of 3 fermions which can occupy any of the 4 available energy states with equal probability. The entropy of the system is

(a) $k_B \ln 2$ (b) $2k_B \ln 2$ (c) $2k_B \ln 4$ (d) $3k_B \ln 4$

Ans: (b)

Solution: Number of ways that 3 fermions will adjust in 4 available energy is ${}^4C_3 = 4$ so entropy is $k_B \ln 4 = 2k_B \ln 2$

GATE 2015

Q30. In Boss-Einstein condensation, the particles

(a) have strong interparticle attraction

(b) condense in real space

(c) have overlapping wavefunctions

(d) have large and positive chemical potential

Ans.: (c)

Solution: In Bose- Einstein condensates, the particles have overlapping wave function.

Q31. For a black body radiation in a cavity, photons are created and annihilated freely as a result of emission and absorption by the walls of the cavity. This is because

(a) the chemical potential of the photons is zero

(b) photons obey Pauli exclusion principle

(c) photons are spin-1 particles

(d) the entropy of the photons is very large

Ans.: (a)

Solution: The chemical potential of photon is zero

Q32. Consider a system of N non-interacting spin $-\frac{1}{2}$ particles, each having a magnetic moment μ, is in a magnetic field $\vec{B} = B\hat{z}$. If E is the total energy of the system, then number of accessible microstates Ω is given by

(a) $\Omega = \dfrac{N!}{\frac{1}{2}\left(N - \frac{E}{\mu B}\right)!\ \frac{1}{2}\left(N + \frac{E}{\mu B}\right)!}$ (b) $\Omega = \dfrac{\left(N - \frac{E}{\mu B}\right)!}{\left(N + \frac{E}{\mu B}\right)!}$

(c) $\Omega = \frac{1}{2}\left(N - \frac{E}{\mu B}\right)!\ \frac{1}{2}\left(N + \frac{E}{\mu B}\right)!$ (d) $\Omega = \dfrac{N!}{\left(N + \frac{E}{\mu B}\right)!}$

Ans.: (a)

Solution: Number of microstate is ${}^{N}C_{n_1}$, where n_1 is number of particle in $+\frac{1}{2}$ state and $n_2 = (N - n_1)$ is number of state in $-\frac{1}{2}$ state.

where $n_1 = \frac{1}{2}\left(N - \frac{E}{\mu B}\right)$, $n_2 = \frac{1}{2}\left(N + \frac{E}{\mu B}\right)$

So, number of microstate $= \dfrac{\lfloor N}{\left\lfloor\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\right\rfloor\left\lfloor\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\right\rfloor}$

Q33. The average energy U of a one dimensional quantum oscillator of frequency ω and in contact with a heat bath at temperature T is given by

(a) $U=\frac{1}{2}\hbar\omega\coth\left(\frac{1}{2}\beta\hbar\omega\right)$ (b) $U=\frac{1}{2}\hbar\omega\sinh\left(\frac{1}{2}\beta\hbar\omega\right)$

(c) $U=\frac{1}{2}\hbar\omega\tanh\left(\frac{1}{2}\beta\hbar\omega\right)$ (d) $U=\frac{1}{2}\hbar\omega\cosh\left(\frac{1}{2}\beta\hbar\omega\right)$

Ans.: (a)

Solution: $\because Z=\sum e^{-\beta E_i}=\sum_{i=0}^{\infty}e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}$ where $E=\left(n+\frac{1}{2}\right)\hbar\omega\Rightarrow Z=\frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}$

$$\because U=\frac{-\partial}{\partial\beta}\ln Z=-\frac{\partial}{\partial\beta}\ln\left[\frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}\right]=\frac{\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right)$$

Q34. The entropy of a gas containing N particles enclosed in a volume V is given by $S=Nk_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)$, where E is the total energy, a is a constant and k_B is the Boltzmann constant. The chemical potential μ of the system at a temperature T is given by

(a) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-\frac{5}{2}\right]$ (b) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-\frac{3}{2}\right]$

(c) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{3/2}}\right)-\frac{5}{2}\right]$ (d) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{3/2}}\right)-\frac{3}{2}\right]$

Ans.: (a)

Solution: $\left(\frac{\partial G}{\partial T}\right)_P=-S=-Nk_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)\ \because S=Nk_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)$

$$\Rightarrow G=-Nk_BT\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)+\ln A$$

$$\Rightarrow\mu=\left(\frac{\partial G}{\partial N}\right)=-\left[k_BT\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)+Nk_BT\frac{N^{5/2}}{aVE^{3/2}}.\frac{(-5/2)}{N^{7/2}}aVE^{3/2}\right]=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{\frac{5}{2}}}\right)-\frac{5}{2}\right]$$

GATE-2016

Q35. The total power emitted by a spherical black body of radius R at a temperature T is P_1. Let P_2 be the total power emitted by another spherical black body of radius $\frac{R}{2}$ kept at temperature $2T$. The ratio, $\frac{P_1}{P_2}$ is _____. (Give your answer upto two decimal places)

Ans.: 0.25

Solution: $P \propto AT^4 \Rightarrow \frac{P_1}{P_2} = \frac{R_1^2 T_1^4}{R_2^2 T_2^4} = \frac{R^2 T^4}{\left(\frac{R}{2}\right)^2 (2T)^4} = \frac{4}{16} = \frac{1}{4} = 0.25$

Q36. The entropy S of a system of N spins, which may align either in the upward or in the downward direction, is given by $S = -k_B N\left[p \ln p + (1-p) In(1-p)\right]$. Here k_B is the Boltzmann constant. The probability of alignment in the upward direction is p. The value of p, at which the entropy is maximum, is _____. (Give your answer upto one decimal place)

Ans.: 0.5

Solution: $S = -k_B N\left[p \ln p + (1-p) In(1-p)\right]$

For maximum entropy, $\frac{dS}{dp} = 0 \Rightarrow \ln p + p \times \frac{1}{p} - \ln(1-p) + (1-p) \times \frac{1}{1-p}(-1) = 0$

$$\ln p + 1 - \ln(1-p) - 1 = 0 \Rightarrow \ln\left(\frac{p}{1-p}\right) = 0 \Rightarrow p = 1-p \Rightarrow p = 0.5$$

Q37. For a system at constant temperature and volume, which of the following statements is correct at equilibrium?

(a) The Helmholtz free energy attains a local minimum.

(b) The Helmholtz free energy attains a local maximum.

(c) The Gibbs free energy attains a local minimum.

(d) The Gibbs free energy attains a local maximum.

Ans.: (a)

Solution: $dF = -SdT - PdV$

Q38. N atoms of an ideal gas are enclosed in a container of volume V. The volume of the container is changed to $4V$, while keeping the total energy constant. The change in the entropy of the gas, in units of $Nk_B \ln 2$, is _____, where k_B is the Boltzmann constant.

Ans.: 2

Solution: $S_1 = -Nk_B \ln 1,\ S_2 = -Nk_B \ln \frac{1}{4} \Rightarrow \Delta S = S_2 - S_1 = Nk_B \ln 4 = 2Nk_B \ln 2$

Q39. Consider a system having three energy levels with energies 0, 2ε and 3ε ,with respective degeneracies of 2,2 and 3 . Four bosons of spin zero have to be accommodated in these levels such that the total energy of the system is $10\ \varepsilon$. The number of ways in which it can be done is _____.

Ans.: 18

Solution: The system have energy $10\ \varepsilon$, if out of four boson two boson are in energy level 2ε and two boson are in energy level 3ε and

$$W = \prod_i \frac{\lfloor n_i + g_i - 1}{\lfloor n_i \rfloor g_i - 1},\ n_1 = 2, g_1 = 2 \text{ and } n_2 = 2, g_2 = 3$$

$$W = \frac{\lfloor 2+2-1}{\lfloor 2 \rfloor 2-1} \times \frac{\lfloor 2+3-1}{\lfloor 2 \rfloor 3-1} = 3 \times 6 = 18$$

Q40. A two-level system has energies zero and E. The level with zero energy is non-degenerate, while the level with energy E is triply degenerate. The mean energy of a classical particle in this system at a temperature T is

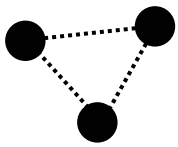
(a) $\frac{Ee^{\frac{-E}{k_B T}}}{1+3e^{\frac{-E}{k_B T}}}$ (b) $\frac{Ee^{\frac{-E}{k_B T}}}{1+e^{\frac{-E}{k_B T}}}$ (c) $\frac{3Ee^{\frac{-E}{k_B T}}}{1+e^{\frac{-E}{k_B T}}}$ (d) $\frac{3Ee^{\frac{-E}{k_B T}}}{1+3e^{\frac{-E}{k_B T}}}$

Ans.: (d)

Solution: $\langle E \rangle = \frac{\sum_i g_i E_i e^{-\frac{E_i}{kT}}}{\sum_i g_i e^{-\frac{E_i}{kT}}} = \frac{0 \times e^{-\frac{0}{kT}} + 3 \times E \times e^{-\frac{E}{kT}}}{e^{-\frac{0}{kT}} + 3 \times e^{-\frac{E}{kT}}} = \frac{3Ee^{\frac{-E}{k_B T}}}{1+3e^{\frac{-E}{k_B T}}}$

GATE 2017

Q41. Consider a triatomic molecule of the shape shown in the figure in three dimensions. The heat capacity of this molecule at high temperature (temperature much higher than the vibrational and rotational energy scales of the molecule but lower than its bond dissociation energies) is:



(a) $\frac{3}{2}k_B$ (b) $3k_B$ (c) $\frac{9}{2}k_B$ (d) $6k_B$

Ans. : (d)

Solution: If given molecules are at lower temperature i.e. atoms are attached to rigid rod then degree of freedom is 6, so internal energy is $\frac{6k_BT}{2}$, but at high temperature, vibration mode will active, so there are three extra vibration mode will active, so total energy

$$U = 3k_BT + 3k_BT = 6k_BT$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 6k_B$$

Q42. A reversible Carnot engine is operated between temperatures T_1 and $T_2\left(T_2 > T_1\right)$ with a photon gas as the working substance. The efficiency of the engine is

(a) $1-\frac{3T_1}{4T_2}$ (b) $1-\frac{T_1}{T_2}$ (c) $1-\left(\frac{T_1}{T_2}\right)^{3/4}$ (d) $1-\left(\frac{T_1}{T_2}\right)^{4/3}$

Ans. : (b)

Solution: Efficiency of Carnot engine does not depends on nature of working substance rather depends on temperature of source and sink

$$\eta = 1-\frac{T_1}{T_2}$$

Q43. Water freezes at 0^0C at atmospheric pressure $\left(1.01\times10^5\,Pa\right)$. The densities of water and ice at this temperature and pressure are $1000\,kg/m^3$ and $934\,kg/m^3$ respectively. The latent heat of fusion is $3.34\times10^5\,J/kg$. The pressure required for increasing the melting temperature of ice by 10^0C is…… GPa. (up to two decimal places)

Ans. : 0.01×10^{-2}

Solution: $\left(\frac{dP}{dT}\right)_V = \frac{L}{T(v_2 - v_1)} \Rightarrow \int_{P_1}^{P_2} dP = \frac{L}{(v_2 - v_1)} \int_{T_1}^{T_2} \frac{dT}{T} \Rightarrow P_2 - P_1 = \frac{L}{(v_2 - v_1)} \ln \frac{T_2}{T_1}$

$\Rightarrow P_2 = P_1 + \frac{L}{(v_2 - v_1)} \ln \frac{T_2}{T_1} = 1 \times 10^5 Pa = 0.01 \times 10^{-2} GPa$

Q44. Consider N non- interacting, distinguishable particles in a two-level system at temperature T. The energies of the levels are 0 and ε, where $\varepsilon > 0$. In the high temperature limit $(k_B T > \varepsilon)$, what is the population of particles in the level with energy ε?

(a) $\frac{N}{2}$ (b) N (c) $\frac{N}{4}$ (d) $\frac{3N}{4}$

Ans. : (a)

Solution: $P(\varepsilon) = \frac{\exp -\frac{\varepsilon}{kT}}{1 + \exp -\frac{\varepsilon}{kT}}$, population of particle in the level with energy ε is

$NP(\varepsilon) = N \frac{\exp -\frac{\varepsilon}{kT}}{1 + \exp -\frac{\varepsilon}{kT}}$, for $(k_B T > \varepsilon)$, $NP(\varepsilon) = N \frac{\exp -\frac{\varepsilon}{kT}}{1 + \exp -\frac{\varepsilon}{kT}} = N \frac{1}{1+1} = \frac{N}{2}$

Q45. The energy density and pressure of a photon gas are given by $u = aT^4$ and $P = \frac{u}{3}$. Where T is the temperature and a is the radiation constant. The entropy per unit volume is given by $\alpha a T^3$. The value of α is…… (up to two decimal places)

Ans. : 1.33

Solution: $TdS = dU + PdV \Rightarrow T\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + P$

$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T} = \frac{aT^4}{T} + \frac{aT^4}{3T} = \frac{4}{3} aT^3 = 1.33$

Q46. Consider two particles and two non-degenerate quantum levels 1 and 2. Level 1 always contains a particle. Hence, what is the probability that level 2 also contains a particle for each of the two cases:

(i) when the two particles are distinguishable and (ii) when the two particles are bosons?

(a) (i) $\frac{1}{2}$ and (ii) $\frac{1}{3}$ (b) (i) $\frac{1}{2}$ and (ii) $\frac{1}{2}$

(c) (i) $\frac{2}{3}$ and (ii) $\frac{1}{2}$ (d) (i) 1 and (ii) 0

Ans. : (c)

Solution: (I): For distinguishable particle: $\frac{B}{A}$ $\frac{A}{B}$ $\frac{}{AB}$, $P(2)=\frac{2}{3}$

(II): For indistinguishable particle (Bosons): $\frac{A}{A}$ $\frac{}{AA}$, $P(2)=\frac{1}{2}$

GATE-2018

Q47. A microcanonical ensemble consists of 12 atoms with each taking either energy 0 state, or energy $\in$ state. Both states are non-degenerate. If the total energy of this ensemble is $4\in$, its entropy will be _____ k_B (up to one decimal place), where k_B is the Boltzmann constant.

Ans. : 6.204

Solution: The number of ways having total energy $4\in$, out of 12 atom is

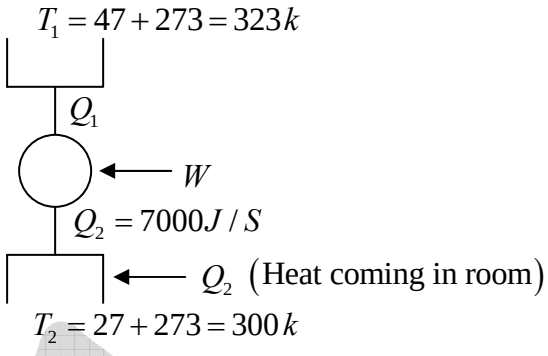
$$= {}^{12}C_4 = \frac{\lfloor 12}{\lfloor 4 \lfloor 8} = \frac{12\times11\times10\times9}{4\times3\times2} = 495$$

Hence, entropy, $S = k_B \ln w = k_B \ln(495) = k_B(6.204) = 6.204\ k_B$

Q48. An air-conditioner maintains the room temperature at $27^0 C$ while the outside temperature is $47^0 C$. The heat conducted through the walls of the room from outside to inside due to temperature difference is 7000 W. The minimum work done by the compressor of the air-conditioner per unit time is_____ W.

Ans. : 466.67

Solution: $Q_2 + W = Q_1$



Coefficient of performance of refrigerator $(AC) = \frac{Q_2}{W}$

Also, coefficient of performance of refrigerator, $= \frac{T_2}{T_1 - T_2}$

$\Rightarrow \frac{300}{47-27} = \frac{7000}{W}$

$\Rightarrow W = \frac{7000 \times 20}{300} J/s = \frac{1400}{3} = 466.67\ W$

Q49. Two solid spheres A and B have same emissivity. The radius of A is four times the radius of B and temperature of A is twice the temperature of B. The ratio of the rate of heat radiated from A to that from B is _____.

Ans. : 256

Solution: $\frac{\text{Rate of heat radiation from solid sphere}(A)}{\text{Rate of heat radiation from solid sphere}(B)} = \frac{4\pi R_A^2 T_A^4}{4\pi R_B^2 T_B^4}$

$\because\ R_A = 4R_B$ and $T_A = 2T_B$

$= \frac{4\pi R_A^2 T_A^4}{4\pi R_B^2 T_B^4} = \frac{(4R_B)^2 \times (2T_B)^4}{(R_B)^2 \times (T_B)^4} = 16 \times 16 = 256$

Q50. The partition function of an ensemble at a temperature T is

$$Z = \left(2\cosh\frac{\varepsilon}{k_B T}\right)^N$$

where k_B is the Boltzmann constant. The heat capacity of this ensemble at $T = \frac{\varepsilon}{k_B}$ is $X\,Nk_B$, where the value of X is _____ (up to two decimal places).

Ans. : 0.42

Solution: The partition function, $z = \left[2\cosh\left(\frac{\varepsilon}{k_B T}\right)\right]^N$

The average energy, $\langle E \rangle = k_B T^2 \frac{\partial(\ln z)}{\partial T}$

$$= \frac{Nk_BT^2\left[2\sinh\left(\frac{\varepsilon}{k_BT}\right)\right]\left(\frac{-\varepsilon}{k_BT^2}\right)}{2\cosh\left(\frac{\varepsilon}{k_BT}\right)} = -N\varepsilon\tanh\left(\frac{\varepsilon}{k_BT}\right)$$

$$C = \frac{d\langle E\rangle}{dT} = -N\varepsilon\sec h^2\left(\frac{\varepsilon}{k_BT}\right)\cdot\left(\frac{-\varepsilon}{k_BT^2}\right)$$

At $T = \frac{\varepsilon}{k}$, $C = \frac{N\varepsilon^2}{k\cdot\left(\varepsilon^2/k^2\right)}\sec h^2(1) = Nk\sec h^2(1) = 0.42Nk_B$

GATE-2019

Q51. Consider a one-dimensional gas of N non-interacting particles of mass m with the Hamiltonian for a single particle given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + 2x\right)$$

The high temperature specific heat in units of $R = Nk_B$ (k_B is the Boltzmann constant) is

(a) 1 (b) 1.5 (c) 2 (d) 2.5

Ans. : (c)

Solution: $\langle H\rangle = \left\langle\frac{p^2}{2m}\right\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle + \frac{1}{2}m\omega^2\langle 2x\rangle = \frac{NkT}{2} + \frac{NkT}{2} + U_0$

$$\langle H\rangle = NkT$$

$$C_V = \frac{\partial H}{\partial T} = NkT$$

Q52. A large number N of ideal bosons, each of mass m, are trapped in a three-dimensional potential $V(r) = \frac{m\omega^2r^2}{2}$. The bosonic system is kept at temperature T which is much lower than the Bose-Einstein condensation temperature T_C. The chemical potential (μ) satisfies

(a) $\mu \le \frac{3}{2}\hbar\omega$ (b) $2\hbar\omega > \mu > \frac{3}{2}\hbar\omega$

(c) $3\hbar\omega > \mu > 2\hbar\omega$ (d) $\mu = 3\hbar\omega$

Ans. : (a)

Q53. At temperature T Kelvin (K), the value of the Fermi function at an energy $0.5eV$ above the Fermi energy is 0.01. Then T, to the nearest integer, is _____

$(k_B = 8.62\times10^{-5} eV/K)$

Ans.: 1262

Solution: $F(E) = \dfrac{1}{e^{(E-E_F)/k_BT}+1} \Rightarrow e^{(E-E_F)/k_BT}+1 = \dfrac{1}{F(E)}$

$$\Rightarrow e^{-(E-E_F)/k_BT} = \frac{1-F}{F} \Rightarrow \frac{E-E_F}{k_BT} = \ln\left(\frac{1-F}{F}\right) \Rightarrow T = \frac{E-E_F}{k_B \ln\left(\dfrac{1-F}{F}\right)}$$

$$\therefore \quad T = \frac{0.5}{8.62\times10^{-5}\ln\left(\dfrac{0.99}{0.01}\right)} = \frac{0.5}{8.62\times\ln(99)}$$

$$= \frac{0.5\times10^5}{8.62\times4.595} = 1262.3\ K$$

Q54. In a thermally insulated container, $0.01\,kg$ of ice at $273\,K$ is mixed with $0.1\,kg$ of water at $300\,K$. Neglecting the specific heat of the container, the change in the entropy of the system in J/K on attaining thermal equilibrium (rounded off to two decimal places) is_____

Ans. : 1.03

Solution: $T_{eq} = 290.29\,K$ (Heat gain = Heat lost)

$$m_{ice}L + m_{ice}C(T-273) = m_\omega C(300-T)$$

$$T = 290.29\,K$$

$$\Delta s = \Delta s_{ice} + \Delta s_{water}$$

$$(\Delta s)_{ice} = \frac{m_{ice}L}{T_{ice}} + m_{ice}C\ln\frac{T_i}{T_{ice}} = 14.85\ J/K$$

$$(\Delta S)_{water} = m_\omega C\ln\frac{290.29}{300} = -13.82\ J/K$$

$$\Delta S = 1.03\ J/K$$

Q55. Consider two system A and B each having two distinguishable particles. In both the systems, each particle can exist in states with energies $0,1,2$ and 3 units with equal probability. The total energy of the combined system is 5 units. Assuming that the system A has energy 3 units and the system B has energy 2 units, the entropy of the system is $k_B \ln \lambda$. The value of λ is_____

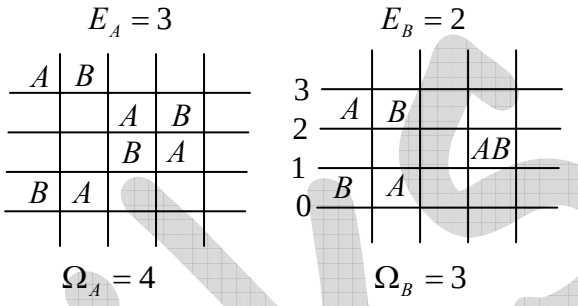
Ans. : 12

Solution:

$\Omega = 4 \times 3 = 12$

$S = \ln \Omega = k_B \ln 12$

$\lambda = 12$.



ELECTRONICS SOLUTIONS

GATE-2010

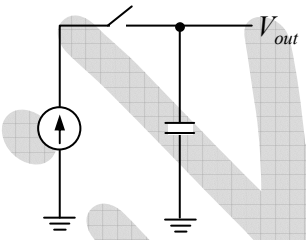
Q1. The voltage resolution of a 12-bit digital to analog converter (DAC), whose output varies from $-10\,V$ to $+10\,V$ is, approximately

(a) $1\,mV$ (b) $5\,mV$ (c) $20\,mV$ (d) $100\,mV$

Ans: (b)

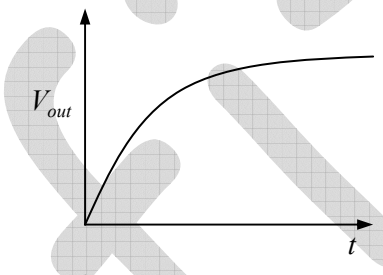
Solution: Voltage resolution$=\dfrac{20V}{2^{12}-1}=4.8\,mV$

Q2. The figure shows a constant current source charging a capacitor that is initially uncharged.

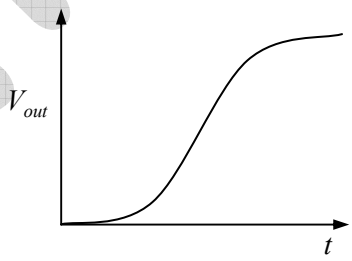


If the switch is closed at $t = 0$, which of the following plots depicts correctly the output voltage of the circuit as a function of time?

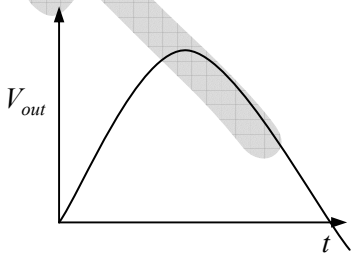
(a)



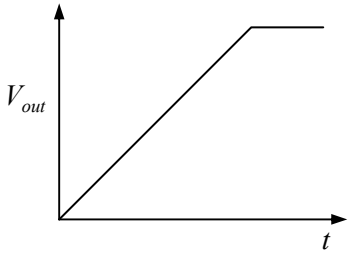
(b)



(c)



(d)

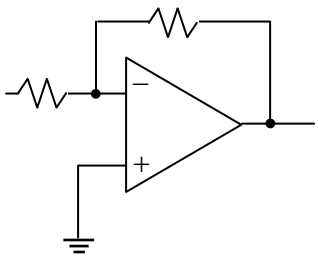


Ans: (d)

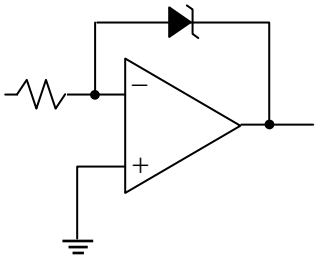
Solution: $I_0=\dfrac{CdV_0}{dt}\Rightarrow V_0=\dfrac{I_0}{C}t$

Q3. In one of the following circuits, negative feedback does not operate for a negative input. Which one is it? The opamps are running from ± 15 V supplies.

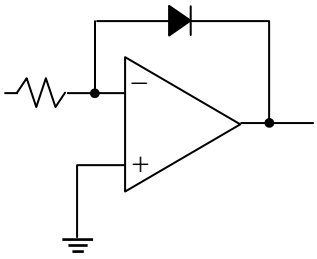
(a)



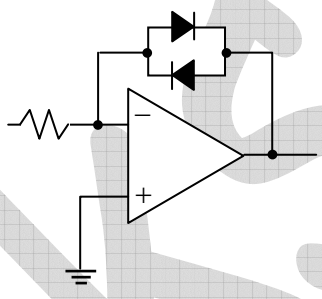
(b)



(c)



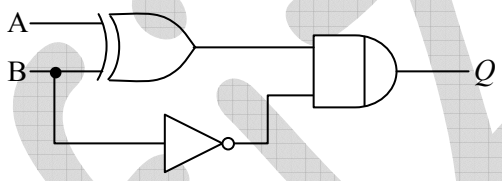
(d)



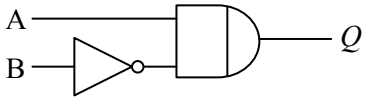
Ans: (c)

Q4. For any set of inputs, A and B, the following circuits give the same output, Q, except one. Which one is it?

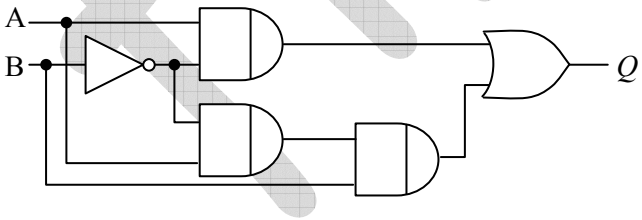
(a)



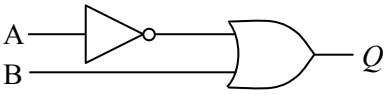
(b)



(c)



(d)



Ans. : (d)

GATE-2011

Q5. Which of the following statements is **CORRECT** for a common emitter amplifier circuit?

(a) The output is taken from the emitter

(b) There is 180^{o} phase shift between input and output voltages

(c) There is no phase shift between input and output voltages

(d) Both *p-n* junctions are forward biased

Ans: (b)

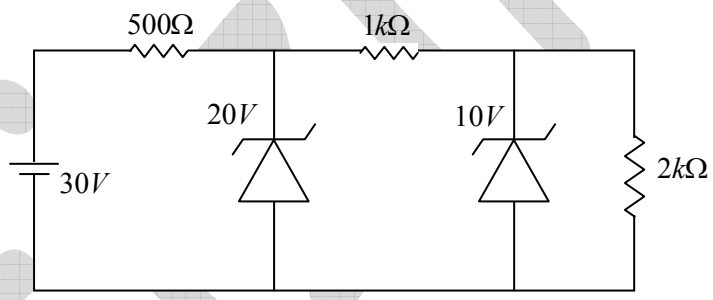
Q6. For an intrinsic semiconductor, m_e^* and m_h^* are respectively the effective masses of electrons and holes near the corresponding band edges. At a finite temperature the position of the Fermi level

(a) depends on m_e^* but not on m_h^* (b) depends on m_h^* but not on m_e^*

(c) depends on both m_e^* and m_h^* (d) depends neither on m_e^* nor on m_h^*

Ans: (c)

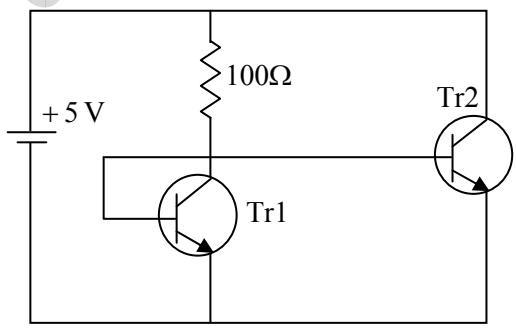
Q7. In the following circuit, the voltage across and the current through the 2 kΩ resistance are



(a) 20 V, 10 mA (b) 20 V, 5 mA (c) 10 V, 10 mA (d) 10 V, 5 mA

Ans: (d)

Q8. In the following circuit, Tr1 and Tr2 are identical transistors having V_{BE} = 0.7 V. The current passing through the transistor Tr2 is



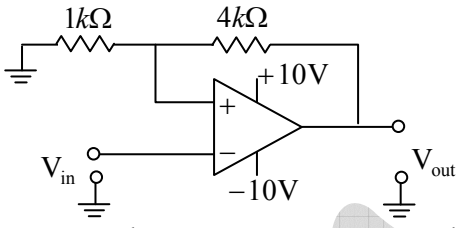
(a) 57 mA (b) 50 mA (c) 48 mA (d) 43 mA

Ans: (d)

Solution: Current through 100Ω, $I = \dfrac{5-0.7}{100} = 43\,mA$

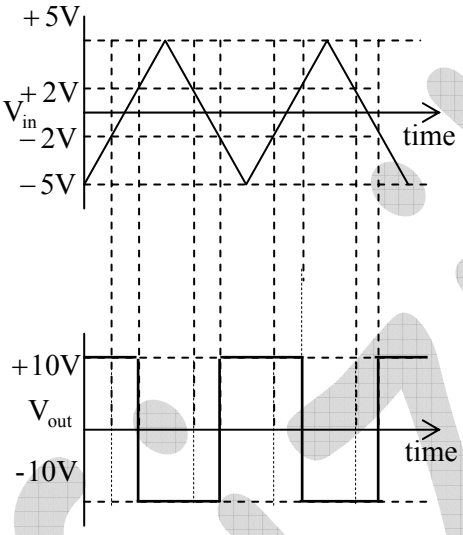
$I = I_C + 2I_B \approx I_C = 43\,mA$.

Q9. Consider the following circuit

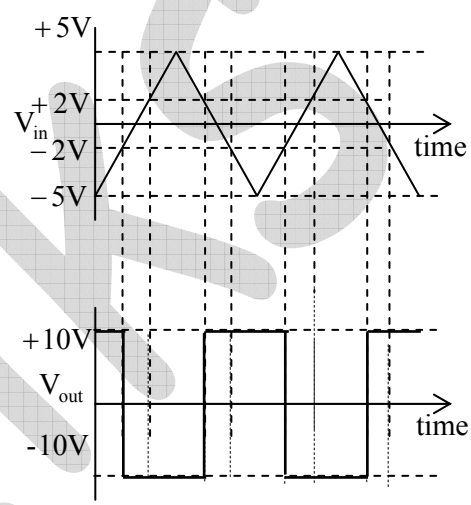


Which of the following correctly represents the output V_{out} corresponding to the input V_{in}?

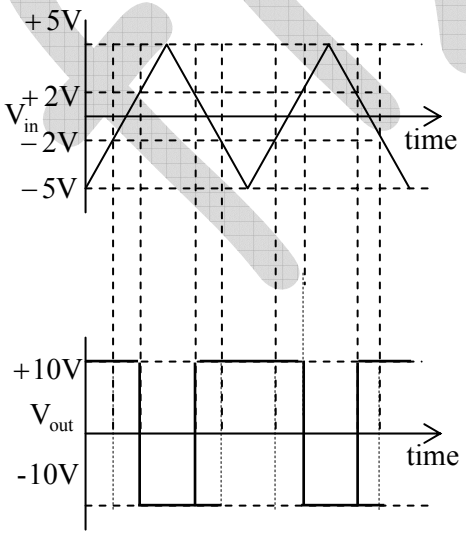
(a)



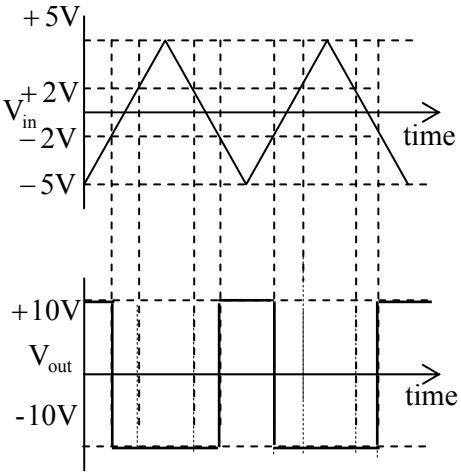
(b)



(c)



(d)



Ans: (a)

Solution: $V_{ut} = \left(\dfrac{1}{1+4}\right)\times 10 = +2V, \quad V_{lt} = \left(\dfrac{1}{1+4}\right)\times -10 = -2V.$

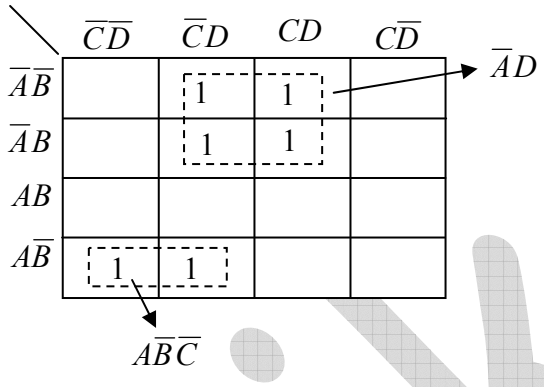
Q10. The following Boolean expression

$Y = A \cdot \overline{B} \cdot \overline{C} \cdot \overline{D} + \overline{A} \cdot B \cdot \overline{C} \cdot D + \overline{A} \cdot \overline{B} \cdot \overline{C} \cdot D + \overline{A} \cdot \overline{B} \cdot C \cdot D + \overline{A} \cdot B \cdot C \cdot D + A \cdot \overline{B} \cdot \overline{C} \cdot D$ can be simplified to

(a) $\overline{A} \bullet \overline{B} \bullet C + A \bullet \overline{D}$ (b) $\overline{A} \bullet B \bullet \overline{C} + A \bullet \overline{D}$

(c) $A \bullet \overline{B} \bullet \overline{C} + \overline{A} \bullet D$ (d) $A \bullet \overline{B} \bullet C + \overline{A} \bullet D$

Ans: (c)



GATE-2012

Q11. If the peak output voltage of a full wave rectifier is 10 V, its d.c. voltage is

(a) 10.0 V (b) 7.07 V (c) 6.36 V (d) 3.18 V

Ans: (c)

Solution: $V_{dc} = \frac{2V_m}{\pi} = \frac{2 \times 10}{22/7} = \frac{14 \times 10}{22} = \frac{70}{11} = 6.36V$

Q12. A *Ge* semiconductor is doped with acceptor impurity concentration of 10^{15} atoms/cm^3. For the given hole mobility of 1800 cm^2/V-s, the resistivity of the material is

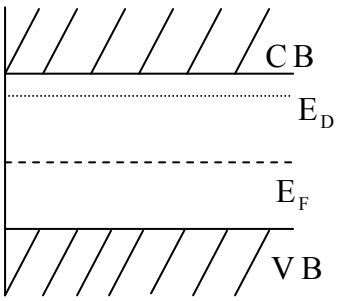
(a) 0.288 Ω cm (b) 0.694 Ω cm (c) 3.472 Ω cm (d) 6.944 Ω cm

Ans: (c)

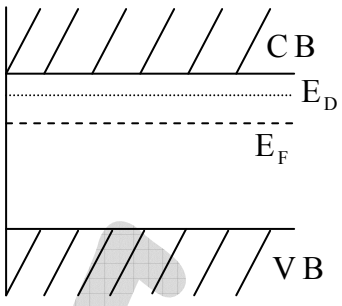
Solution: $\rho = \frac{1}{\sigma} = \frac{1}{N_A e u_h} = \frac{1}{10^{15} \times 1.6 \times 10^{-19} \times 1800} = 3.47 \Omega cm$

Q13. Identify the CORRECT energy band diagram for silcon doped with Arsenic. Here CB, VB, E_D and E_F are conduction band, valence band, impurity level and Fermi level, respectively.

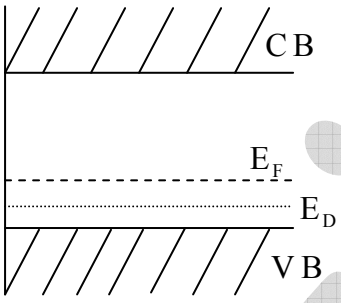
(a)



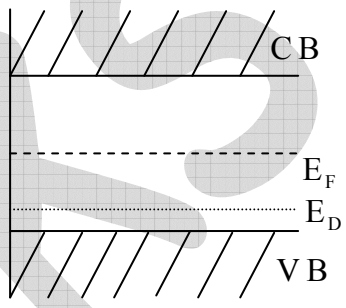
(b)



(c)



(d)



Ans: (b)

Solution: N-type material (*Si* doped with A_S).

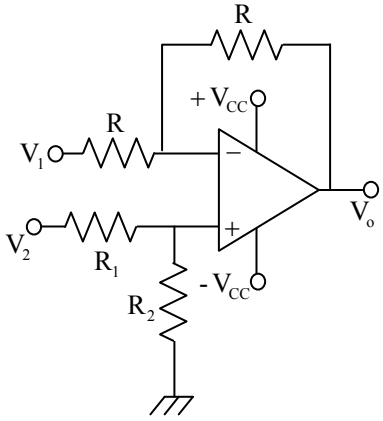
Q14. In the following circuit, for the output voltage to be $V_0 = (-V_1 + V_2/2)$ the ratio R_1/R_2 is

(a) 1/2

(b) 1

(c) 2

(d) 3



Ans: (d)

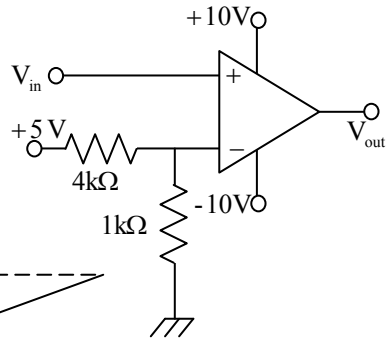
Solution: When $V_2 = 0, \quad v_{01} = -V_1$

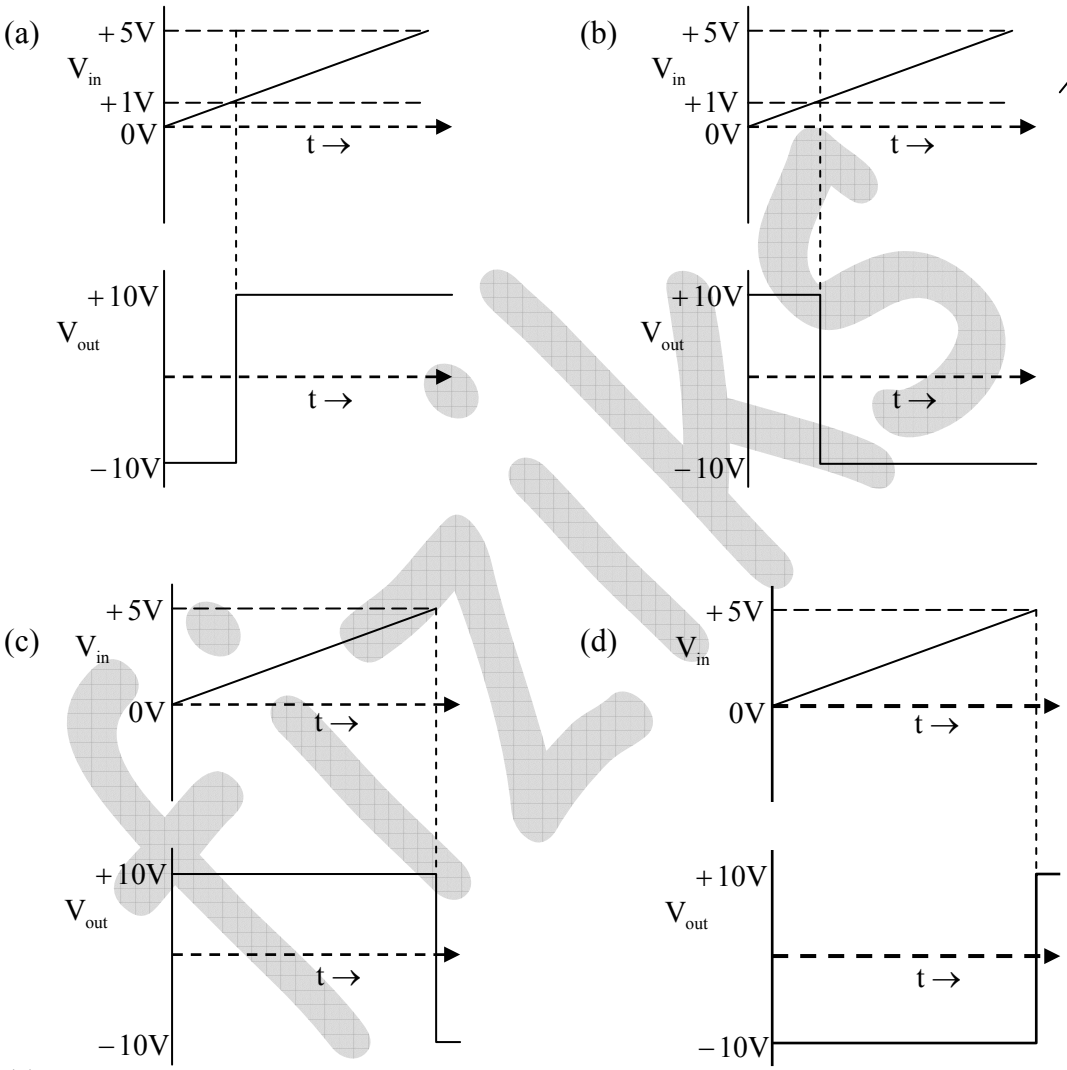
when $V_1 = 0, \quad v_{02} = \left(1 + \frac{R}{R}\right)\left(\frac{R_2}{R_1 + R_2}\right)V_2$

Since $V_0 = -V_1 + \frac{V_2}{2} \Rightarrow 2 \cdot \frac{R_2}{R_1 + R_2} = \frac{1}{2} \Rightarrow \frac{R_1}{R_2} = 3$

Q15. Consider the following OP-AMP circuit.

Which one of the following correctly represents the output V_{out} corresponding to the input V_{in}?



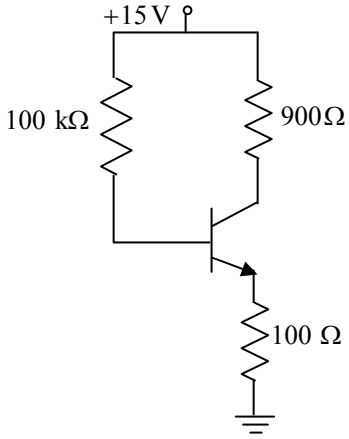


Ans: (a)

Solution: Voltage at inverting input $V_2 = \left(\frac{1}{1+4}\right)\times 5 = +1V.$

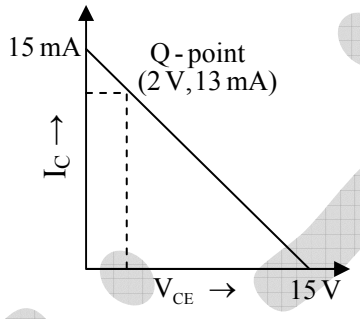
When $v_{in} > +1V$, $v_0 = +V_{CC}$ and when $v_{in} < +1V$, $v_0 = -V_{CC}$

Q16. Consider the following circuit in which the current gain β_{dc} of the transistor is 100.

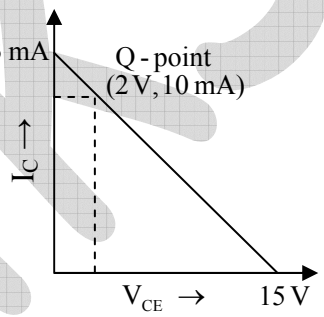


Which one of the following correctly represents the load line (collector current I_C with respect to collector-emitter voltage V_{CE}) and Q-point of this circuit?

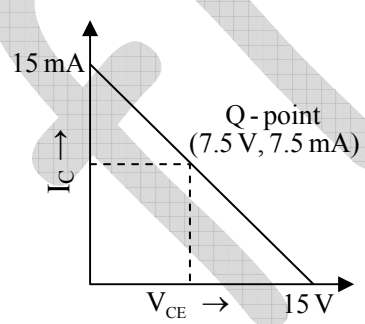
(a)



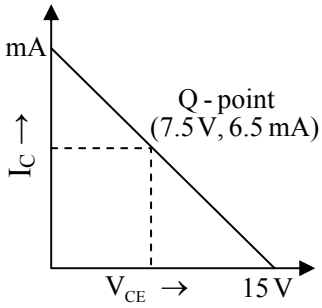
(b)



(c)



(d)



Ans: (a)

Solution: $I_B = \dfrac{V_{CC} - V_{BE}}{R_B + R_E} = \dfrac{15 - 0.7}{100 \times 10^3 + 100} \approx \dfrac{14.3}{100} mA.$

$I_C \approx \beta I_B \approx 14.3mA \approx 13mA$, $V_{CE} = V_{CC} - I_C(R_C + R_E) = 15 - (900 + 100) \times 13 \times 10^{-3} = 2V.$

$I_{C,Sat} = \dfrac{V_{CC}}{R_C + R_E} = \dfrac{15}{1000} = 15\,mA.$

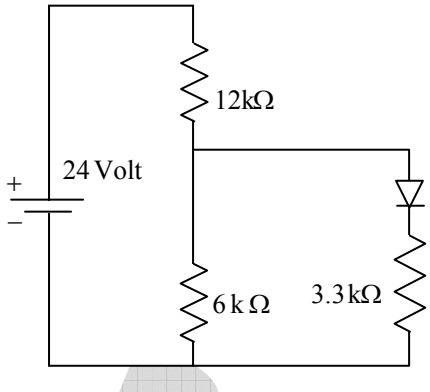
Q17. In the following circuit, the voltage drop across the ideal diode in forward bias condition is 0.7V. The current passing through the diode is

(a) 0.5 mA

(b) 1.0 mA

(c) 1.5 mA

(d) 2.0 mA



Ans: (b)

Solution: Let current through $12k\Omega$ is I and through diode is I_D

Then $0.7 + I_D \times 3.3 = (I - I_D) \times 6$ (1)

and $-24 + I \times 12 + (I - I_D) \times 6 = 0$ (2)

From (1) and (2) $I_D \approx 1mA.$

GATE-2013

Q18. What should be the clock frequency of a $6-bit\ A/D$ converter so that its maximum conserved time is $32\mu s$?

(a) 1 MHz (b) 2 MHz (c) 0.5 MHz (d) 4 MHz

Ans: (c)

Q19. A phosphorous doped silicon semiconductor (doping density: $10^{17}/cm^3$) is heated from 100°C to 200°C. Which one of the following statements is CORRECT?

(a) Position of Fermi level moves towards conduction band

(b) Position of dopant level moves towards conduction band

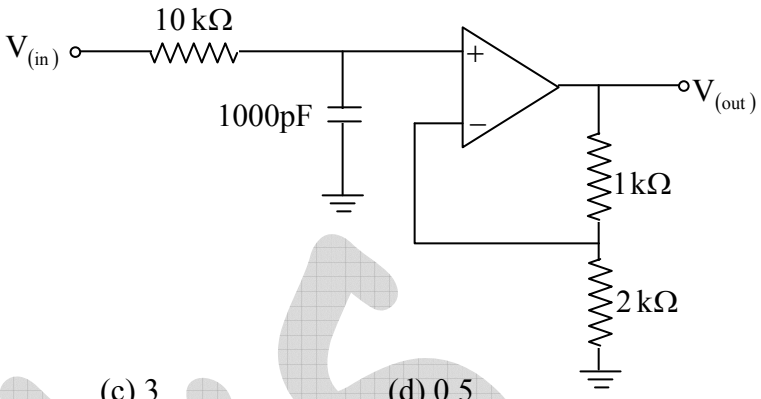
(c) Position of Fermi level moves towards middle of energy gap

(d) Position of dopant level moves towards middle of energy gap

Ans: (c)

Statement for Linked Answer Questions 20 and 21:

Consider the following circuit



Q20. For this circuit the frequency above which the gain will decrease by 20 dB per decade is

(a) $15.9\,kHz$ (b) $1.2kHz$

(c) $5.6\,kHz$ (d) $22.5\,kHz$

Ans: (a)

Solution: $f_H = \dfrac{1}{2\pi RC} = 16kHz$

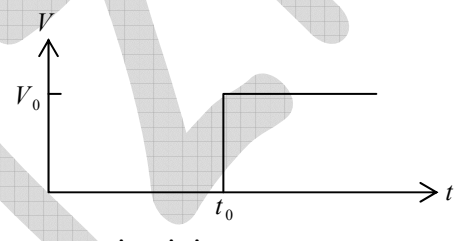
Q21. At $1.2\,kHz$ the closed loop gain is

(a) 1 (b) 1.5 (c) 3 (d) 0.5

Ans: (b) $\left|\dfrac{v_0}{v_{in}}\right| = \dfrac{\left(1+R_F/R_1\right)}{\sqrt{1+\left(f/f_H\right)^2}} = 1.5$

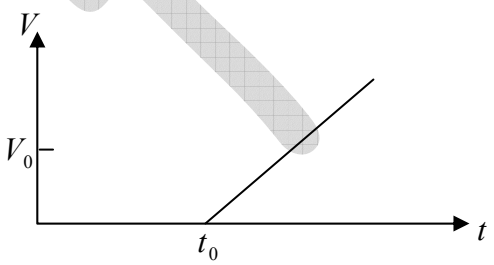
GATE-2014

Q22. The input given to an ideal OP-AMP integrator circuit is

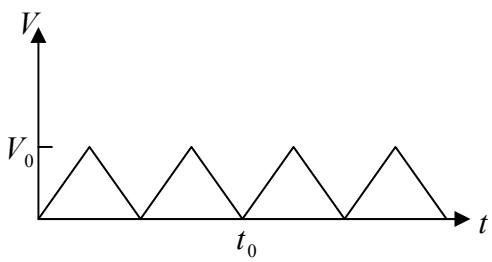


The correct output of the integrator circuit is

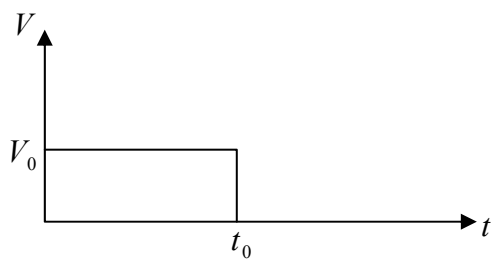
(a)



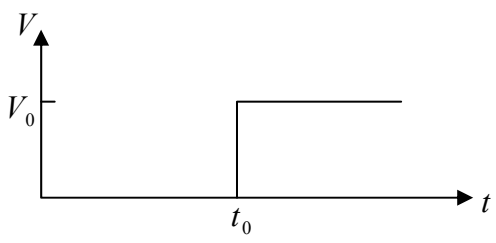
(b)



(c)



(d)



Ans: (a)

Q23. The minimum number of flip-flops required to construct a mod-75 counter is _____

Ans: 7

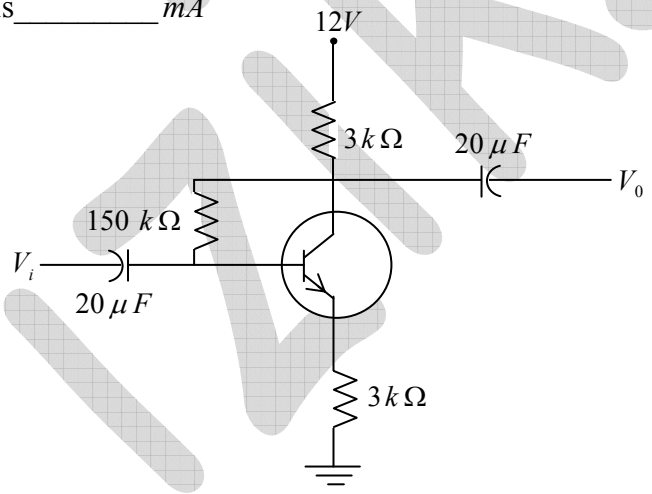
Q24. The donor concentration in a sample of n-type silicon is increased by a factor of 100. The shift in the position of the Fermi level at 300K, assuming the sample to non degenerate is _____ meV. $\left(k_B T = 25meV\ at\ 300K\right)$

Ans: 115.15

Solution: $E_C - E_F = kT\ln\left(\frac{N_c}{N_d}\right)$ and $E_C - E_F' = kT\ln\left(\frac{N_c}{100N_d}\right) = kT\ln\left(\frac{N_c}{N_d}\right) - kT\ln(100)$

Thus shift is $\Delta E = kT\ln(100) = 25\ln(100)meV = 115.15\ meV$

Q25. The current gain of the transistor in the following circuit is $\beta_{dc} = 100$. The value of collector current I_C is_____ mA



Ans: 1.6

Solution: $I_B = \frac{V_{CC} - V_{BE}}{R_B + \beta\left(R_C + R_E\right)} = \frac{12 - 0}{150 + 100\left(3 + 3\right)} = 0.016\ mA \Rightarrow I_C = \beta I_B = 1.6\ mA$

Q26. In order to measure a maximum of $1V$ with a resolution of $1mV$ using a $n-$ bit $\frac{A}{D}$ converter working under the principle of ladder network the minimum value of n is_____

Ans: 10

Solution: $1\times10^{-3} = \frac{1}{2^n - 1} \Rightarrow n \approx 10$

Q27. A low pass filter is formed by a resistance R and a capacitance C. At the cut-off angular frequency $\omega_C = \frac{1}{RC}$ the voltage gain and the phase of the output voltage relative to the input voltage respectively are

(a) 0.71 and 45^o (b) 0.71 and -45^o (c) 0.5 and -90^o (d) 0.5 and 90^o

Ans: (b)

Solution: $\frac{v_0}{v_{in}} = \frac{X_C}{R+X_C} = \frac{1}{\frac{R}{X_C}+1} = \frac{1}{1+j\omega CR}$

At $\omega = \omega_C = \frac{1}{RC} \Rightarrow \frac{v_0}{v_{in}} == \frac{1}{1+j} = \frac{1}{\sqrt{2}e^{j45^0}} = \frac{1}{\sqrt{2}}e^{-j45^0}$

GATE-2015

Q28. The band gap of an intrinsic semiconductor is $E_g = 0.72\,eV$ and $m_h^* = 6m_n^*$. At $300\,K$, the Fermi level with respect to the edge of the valence band (in eV) is at _____(upto three decimal places) $k_B = 1.38\times10^{-23}\,JK^{-1}$

Ans.: 0.395

Solution: $E_i = \frac{E_c+E_v}{2} + \frac{3}{4}kT\ln\left(\frac{m_h^*}{m_n^*}\right)$

$$n_i = N_V e^{-(E_i-E_v)/kT} = \sqrt{N_c N_v}e^{-E_g/2kT} \Rightarrow e^{-(E_i-E_v)/kT} = \sqrt{\frac{N_c}{N_v}}e^{-E_g/2kT} \Rightarrow e^{(E_i-E_v)/kT} = \sqrt{\frac{N_v}{N_c}}e^{E_g/2kT}$$

$$\frac{E_i-E_v}{kT} = \ln\left(\sqrt{\frac{N_v}{N_c}}\right) + \frac{E_g}{2kT} = \ln\left(\frac{m_h^*}{m_e^*}\right)^{\frac{3}{4}} + \frac{E_g}{2kT} \Rightarrow E_i - E_v = \frac{3}{4}kT\ln(6) + \frac{E_g}{2}$$

$$\Rightarrow E_i - E_v = \frac{3}{4}\times0.026\times1.7917 + \frac{0.72}{2} = 0.3949eV \approx 0.395\,eV$$

Q29. Which one of the following DOES NOT represent an exclusive OR operation for inputs A and B?

(a) $(A+B)\overline{AB}$ (b) $A\overline{B}+B\overline{A}$ (c) $(A+B)(\overline{A}+\overline{B})$ (d) $(A+B)AB$

Ans.: (d)

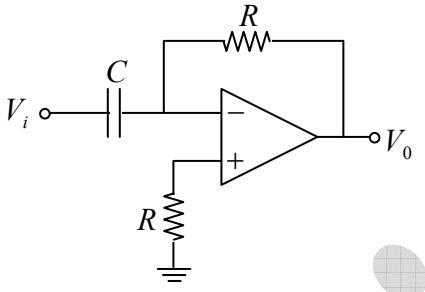
Solution: (a) $(A+B)\overline{AB} = (A+B)(\overline{A}+\overline{B}) = A\overline{B}+\overline{A}B$

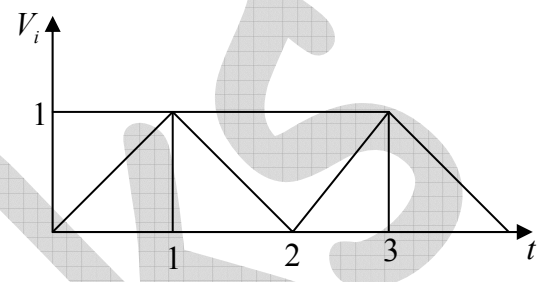
(b) $A\overline{B}+\overline{A}B$

(c) $A\overline{B}+\overline{A}B$

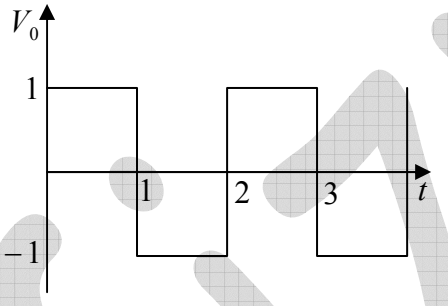
(d) $\left(A+B\right)AB = AB$

Q30. Consider the circuit shown in the figure, where $RC = 1$. For an input signal V_i shown below, choose the correct V_0 from the options:

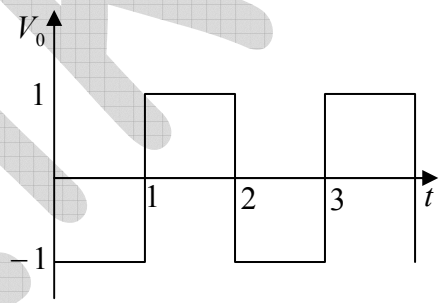




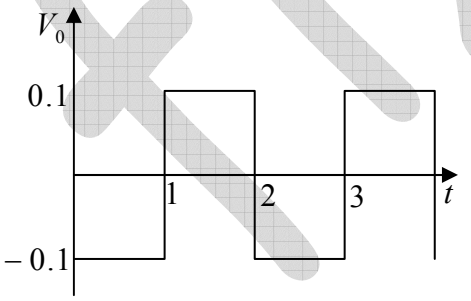
(a)



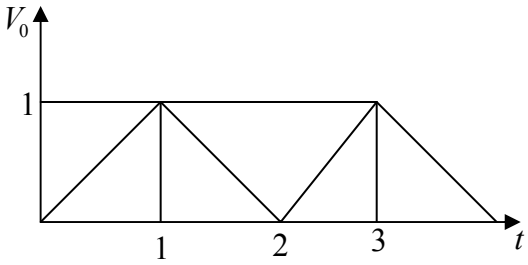
(b)



(c)



(d)

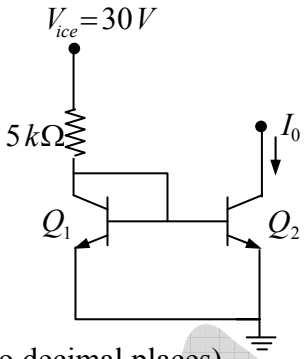


Ans.: (b)

Solution: $C\dfrac{dv_i}{dt} = \dfrac{0-v_0}{R} \Rightarrow v_0 = -RC\dfrac{dv_{in}}{dt} = -\dfrac{dv_{in}}{dt} \Rightarrow v_{in} = -v_0 t$

$v_{in} = +t \Rightarrow v_0 = -1V$ and $v_{in} = -t \Rightarrow v_0 = +1V$

Q31. In the simple current source shown in the figure, Q_1 and Q_2 are identical transistors with current gain $\beta = 100$ and $V_{BE} = 0.7V$

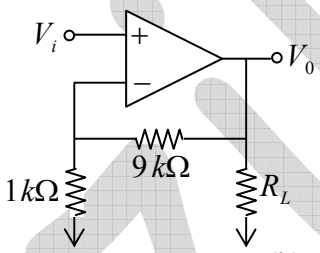


The current $I_0(\text{in } mA)$ is _____ (upto two decimal places)

Ans.: 5.86

Solution: $-V_{CC} + I_C R_C + V_{BE} = 0$, $I_C = \dfrac{30 - 0.7}{5} = \dfrac{29.3}{5} = 5.86\ mA$

Q32. In the given circuit, if the open loop gain $A = 10^5$ the feedback configurations and the closed loop gain A_f are



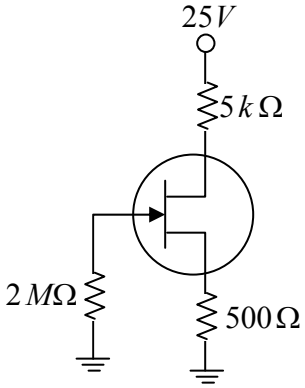
(a) series-shunt, $A_f = 9$ (b) series-series, $A_f = 10$

(c) series-shunt, $A_f = 10$ (d) shunt-shunt, $A_f = 10$

Ans.: (c)

Solution: $A_F = \left(1 + \dfrac{R_F}{R_1}\right) = (1+9) = 10.$

Q33. In the given circuit, the voltage across the source resistor is $1\ V$. The drain voltage (in V) is _____



Ans.: 15

Solution: $V_S = I_D R_S \Rightarrow I_D = \dfrac{1}{500} A \Rightarrow V_D = V_{DD} - I_D R_D = 25 - \dfrac{1}{500} \times 5000 \Rightarrow V_D = 15V$

GATE-2016

Q34. The number density of electrons in the conduction band of a semiconductor at a given temperature is $2\times10^{19}\ m^{-3}$. Upon lightly doping this semiconductor with donor impurities, the number density of conduction electrons at the same temperature becomes $4\times10^{20}\ m^{-3}$. The ratio of majority to minority charge carrier concentration is_____.

Ans : 400

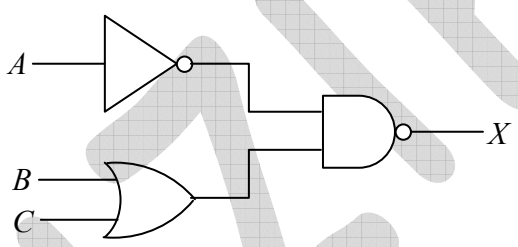
Solution: Intrinsic carrier concentration is $n_i = 2\times10^{19}\ m^{-3}$

Majority carrier concentration is $n = 4\times10^{20}\ \ m^{-3}$

Minority carrier concentration is $p = \dfrac{n_i^2}{n} = \dfrac{\left(2\times10^{19}\right)^2}{4\times10^{20}} = 10^{18}\ \ m^{-3}$

The ratio of majority to minority charge carrier concentration is $\dfrac{n}{p} = \dfrac{4\times10^{20}}{10^{18}} = 400$

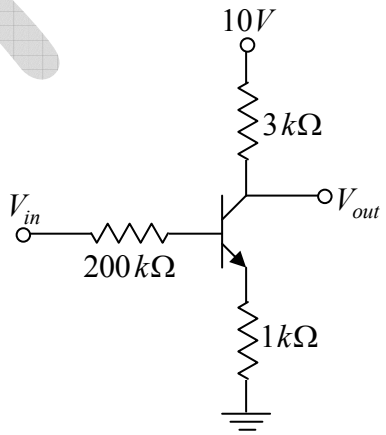
Q35. For the digital circuit given below, the output X is



(a) $\overline{\overline{A}+B.C}$ (b) $\overline{\overline{A}.(B+C)}$ (c) $\overline{A}.(B+C)$ (d) $A+\overline{(B.C)}$

Ans.: (b)

Q36. For the transistor shown in the figure, assume $V_{BE} = 0.7V$ and $\beta_{dc} = 100$. If $V_{in} = 5V, V_{out}$ (in Volts) is _____. (Give your answer upto one decimal place)



Ans.: 5.7

Solution: $I_B = \dfrac{V_{in} - V_{BE}}{R_B + \beta R_E} = \dfrac{5 - 0.7}{200 + 100} = \dfrac{4.3}{300} mA$, $I_C = \beta I_B = 1.433\ mA$

$V_{out} = V_{CC} - I_C R_C \Rightarrow V_{out} = 10 - 1.433 \times 3 = 5.7\ V$

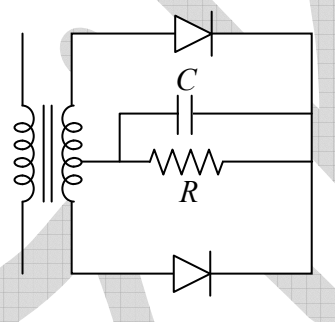
GATE-2017

Q37. The best resolution that a 7 bit A/D convertor with $5V$ full scale can achieve is…… mV . (up to two decimal places)

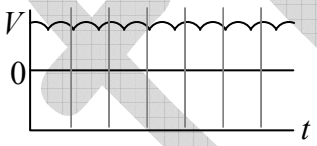
Ans. : 39.37

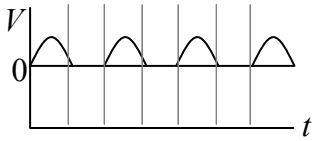
Solution: Resolution $= \dfrac{5}{2^7 - 1} = 39.37\ mV$

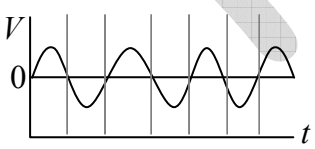
Q38. In the figure given below, the input to the primary of the transformer is a voltage varying sinusoidally with time. The resistor R is connected to the centre tap of the secondary.

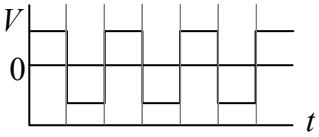


Which one of the following plots represents the voltage across the resistor R as a function of time?

(a) 

(b) 

(c) 

(d) 

Ans. : (a)

Solution:

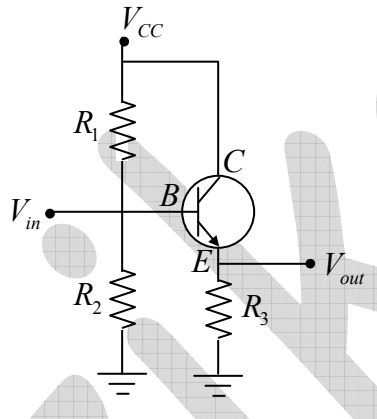
Full wave rectifier with RC filter.

Q39. The minimum number of NAND gates required to construct an OR gate is:

(a) 2 (b) 4 (c) 5 (d) 3

Ans. : (d)

Q40. For the transistor amplifier circuit shown below with $R_1 = 10\,k\Omega, R_2 = 10\,k\Omega, R_3 = 1\,k\Omega$, and $\beta - 99$. Neglecting the emitter diode resistance, the input impedance of the amplifier looking into the base for small ac signal is……$k\Omega$. (up to two decimal places)

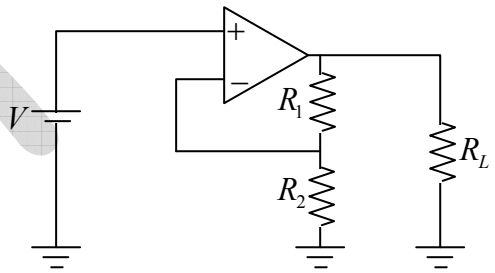


Ans. : 4.75

Solution: $Z_i = Z_b \,||\, R'$ where $Z_b \approx \beta R_3 = 99k\Omega$ and $R' = R_1 \,||\, R_2 = 5k\Omega$

$\Rightarrow Z_i = Z_b \,||\, R' = 4.75k\Omega$

Q41. Consider an ideal operational amplifier as shown in the figure below with $R_1 = 5\,k\Omega, R_2 = 1k\Omega, R_L = 100\,k\Omega$. For an applied input voltage $V = 10\,mV$, the current passing through R_2 is…….μA. (up to two decimal places)



Ans. : 10.0

Solution: $I_2 = \dfrac{V}{R_2} = \dfrac{10}{1} = 10\mu A$

GATE – 2018

Q42. The logic expression $\bar{A}BC + \bar{A}\bar{B}C + AB\bar{C} + A\bar{B}\bar{C}$ can be simplified to

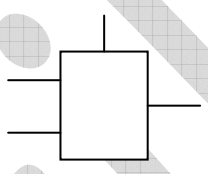
(a) A XOR C (b) A AND C (c) 0 (d) 1

Ans. : (a)

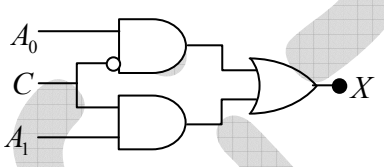
Solution: $Y = \bar{A}BC + \bar{A}\bar{B}C + AB\bar{C} + A\bar{B}\bar{C} = \bar{A}C\left(B + \bar{B}\right) + A\bar{C}\left(B + \bar{B}\right)$

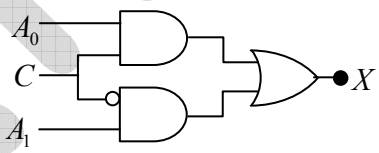
$\Rightarrow Y = \bar{A}C + A\bar{C} = A$ XOR C

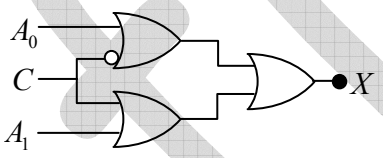
Q43. In a 2-to-1 multiplexer as shown below, the output $X = A_0$ if $C = 0$ and $X = A_1$ if $C = 1$.

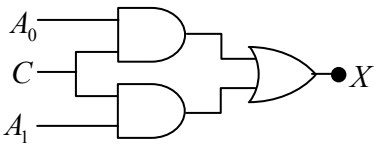


Which one of the following is the correct implementation of this multiplexer?

(a) 

(b) 

(c) 

(d) 

Ans. : (a)

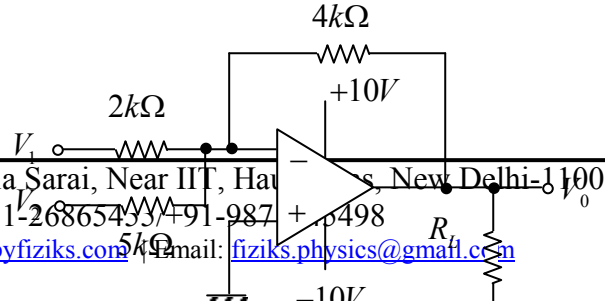
Solution: Check option (a),

$X = A_0\bar{C} + A_1C$

If $C = 0 \Rightarrow X = A_0$

If $C = 1 \Rightarrow X = A_1$

Q44. For an operational amplifier (ideal) circuit shown below,



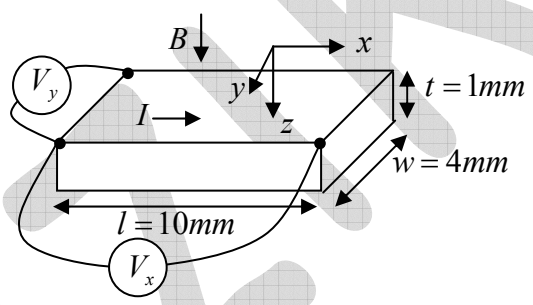
If $V_1 = 1V$ and $V_2 = 2V$, the value of V_0 is _____ V (up to one decimal place).

Ans. : -3.6

Solution: $V_0 = V_{01} + V_{02} = -\frac{4}{2} \times 1V - \frac{4}{5} \times 2V$

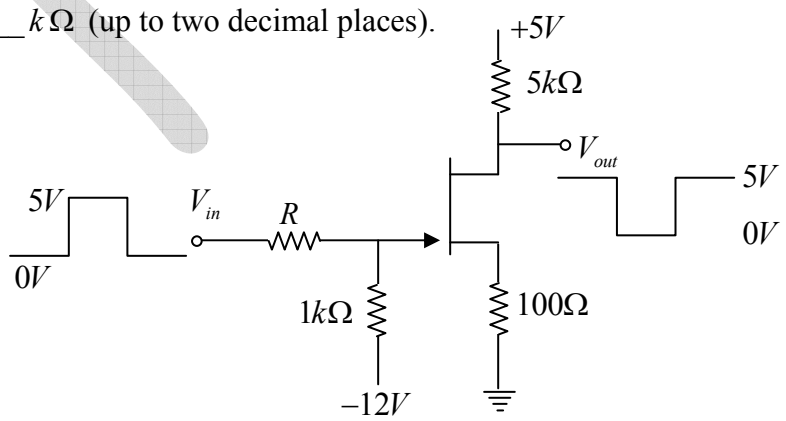
$V_0 = -2 - 1.6 = -3.6V$

Q45. A p - doped semiconductor slab carries a current $I = 100\,mA$ in a magnetic field $B = 0.2\,T$ as shown. One measures $V_y = 0.25\ mV$ and $V_x = 2\,mV$. The mobility of holes in the semiconductor is_____ $m^2V^{-1}s^{-1}$ (up to two decimal places)



Ans. : 1.55

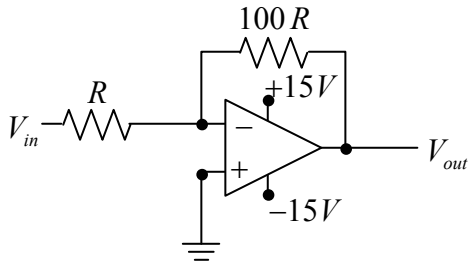
Q46. An n - channel FET having Gate-Source switch-off voltage $V_{GS(OFF)} = -2V$ is used to invert a $0-5V$ square-wave signal as shown. The maximum allowed value of R would be _____ $k\Omega$ (up to two decimal places).



Ans. : 0.70

GATE-2019

Q47. For the following circuit, what is the magnitude of V_{out} if $V_{in} = 1.5V$?



(a) $0.015V$ (b) $0.15V$ (c) $15V$ (d) $150V$

Ans. : (c)

Solution: $V_{out} = -\dfrac{100R}{R} \times 1.5 = -150V \Rightarrow |V_0| = 15V$

Q48. Consider the following Boolean expression:

$$\left(\overline{A} + \overline{B}\right)\left[\overline{A(B+C)}\right] + A\left(\overline{B} + \overline{C}\right)$$

It can be represented by a single three-input logic gate. Identify the gate

(a) AND (b) OR (c) XOR (d) NAND

Ans. : (d)

Solution: $Y = \left(\overline{A} + \overline{B}\right)\left[\overline{A(B+C)}\right] + A\left(\overline{B} + \overline{C}\right)$

$$Y = \left(\overline{A} + \overline{B}\right)\left[\overline{A} + \overline{(B+C)}\right] + A\overline{B} + A\overline{C}$$

$$= \left(\overline{A} + \overline{B}\right)\left[\overline{A} + \overline{B}\,\overline{C}\right] + A\overline{B} + A\overline{C}$$

$$= \overline{A} + \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B} + \overline{B}\,\overline{C} + A\overline{B} + A\overline{C}$$

$$= \overline{A} + \overline{A}\,\overline{B}\,\overline{C} + \overline{B}\,\overline{C} + \overline{A}\,\overline{B} + A\overline{B} + A\overline{C}$$

$$= \overline{A} + \overline{B}\,\overline{C} + \overline{B} + A\overline{C} = \overline{A} + \overline{B} + A\overline{C}$$

$$= \left(\overline{A} + A\overline{C}\right) + \overline{B} = \left(A + \overline{\overline{A}\,\overline{C}}\right) + \overline{B} = \overline{A} + \overline{C} + \overline{B}$$

$$Y = \overline{ABC}$$

Q49. A 3- bit analog-to-digital converter is designed to digitize analog signals ranging from $0V$ to $10V$. For this converter, the binary output corresponding to an input of $6\ V$ is

(a) 011 (b) 101 (c) 100 (d) 010

Ans. : (c)

Solution: $0 \to (000) \to 0V$

$1 \to (001) \to \frac{10}{7} = 1.42V$

$2 \to (010) \to \frac{20}{7} = 2.8V$

$3 \to (011) \to \frac{30}{7} = 4.28V$

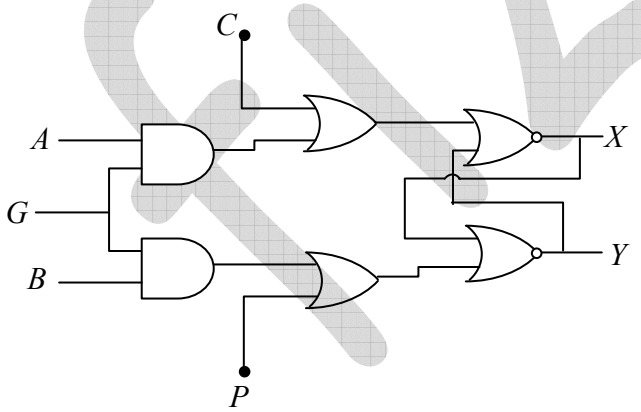
$4 \to (100) \to \frac{40}{7} = 5.71V$

$5 \to (101) \to \frac{50}{7} = 7.14V$

$6 \to (110) \to \frac{60}{7} = 8.57V$

$7 \to (111) \to \frac{70}{7} = 10V$

Q50. For the following circuit, the correct logic values for the entries X_2 and Y_2 in the truth table are



G	A	B	P	C	X	Y
1	0	1	0	0	0	1
0	0	0	1	0	X_2	Y_2
1	0	0	0	1	0	1

(a) 1 and 0 (b) 0 and 0 (c) 0 and 1 (d) 1 and 1

Ans. : (a)

ATOMIC AND MOLECULAR PHYSICS SOLUTIONS

GATE-2010

Q1. To detect trace amounts of gaseous species in a mixture of gases, the preferred probing tool is

(a) Ionization spectroscopy with *X*-rays (b) NMR spectroscopy

(c) ESR spectroscopy (d) Laser spectroscopy

Ans: (a)

Q2. A collection of N atoms is exposed to a strong resonant electromagnetic radiation with N_g atoms in the ground state and N_e atoms in the excited state, such that $N_g+N_e=N$. This collection of two-level atoms will have the following population distribution:

(a) $N_g << N_e$ (b) $N_g >> N_e$

(c) $N_g \approx N_e \approx N/2$ (d) $N_g - N_e \approx N/2$

Ans: (c)

Solution: In two level lair population inversion is possible to achieve at any power level. The maximum possible situation can be $N_g \approx N_e \approx \frac{N}{2}$

Q3. Two states of an atom have definite parities. An electric dipole transition between these states is

(a) Allowed if both the sates have even parity

(b) Allowed if both the states have odd parity

(c) Allowed if the two states have opposite parities

(d) Not allowed unless a static electric field is applied

Ans: (c)

Q4. The spectrum of radiation emitted by a black body at a temperature 1000 K peaks in the

(a) Visible range of frequencies (b) Infrared range of frequencies

(c) Ultraviolet range of frequencies (d) Microwave range of frequencies

Ans: (a)

Q5. The three principal moments of inertia of a methanol (CH_3OH) molecule have the property $I_x = I_y = I$ and $I_z \neq I$. The rotation energy eigenvalues are

(a) $\frac{\hbar^2}{2I}l(l+1)+\frac{\hbar^2 m_1^2}{2I}\left(\frac{1}{I_z}-\frac{1}{I}\right)$ (b) $\frac{\hbar^2}{2I}l(l+1)$

(c) $\frac{\hbar^2 m_1^2}{2I}\left(\frac{1}{I_z}-\frac{1}{I}\right)$ (d) $\frac{\hbar^2}{2I}l(l+1)+\frac{\hbar^2 m_1^2}{2}\left(\frac{1}{I_z}+\frac{1}{I}\right)$

Ans: (a)

Solution: CH_3OH is example of symmetric rotator, where $I_x = I_y \neq I_z$, ($I_x + I_y = I$ and $I_z \neq I$)

The classical expression for energy is $E = \frac{1}{2I}\left(J_x^2 + J_y^2\right)+\frac{1}{2I_z}J_z^2$.

This can be expressed in term of $J^2 = J_x^2 + J_y^2 + J_z^2$ by adding and subtracting J_z^2

$$E = \frac{1}{2I}J^2 + \left(\frac{1}{2I_z}-\frac{1}{2I}\right)J_z^2.$$

Quantum mechanically $E = \frac{\hbar^2}{2I}J(J+1)+\frac{\hbar^2 m_J^2}{2}\left(\frac{1}{I_z}-\frac{1}{I}\right)$

Q6. Match the typical spectra of stable molecules with the corresponding wave-number range

1. Electronic spectra (i) 10^6 cm^{-1} and above
2. Rotational spectra (ii) $10^5 - 10^6$ cm^{-1}
3. Molecule dissociation (iii) $10^8 - 10^2$ cm^{-1}

(a) 1 – ii, 2 – i, 3 – iii (b) 1 – ii, 2 – iii, 3 – i

(b) 1 – iii, 2 – ii, 3 – i (d) 1 – i, 2 – ii, 3 – iii

Ans: (b)

Q7. Consider the operations $P: \vec{r} \to -\vec{r}$ (parity) and T: $t \to -t$ (time reversal). For the electric and magnetic fields $\vec{E}$ and $\vec{B}$, which of the following set of transformations is correct?

(a) $P: \vec{E} \to -\vec{E}, \vec{B} \to \vec{B}$;
$T: \vec{E} \to \vec{E}, \vec{B} \to -\vec{B}$

(b) $P: \vec{E} \to \vec{E}, \vec{B} \to \vec{B}$;
$T: \vec{E} \to \vec{E}, \vec{B} \to \vec{B}$

(c) $P: \vec{E} \to -\vec{E}, \vec{B} \to \vec{B}$;
$T: \vec{E} \to -\vec{E}, \vec{B} \to -\vec{B}$

(d) $P: \vec{E} \to \vec{E}, \vec{B} \to -\vec{B}$;
$T: \vec{E} \to -\vec{E}, \vec{B} \to \vec{B}$

Ans: (b)

Common Data Questions 8 and 9:

In the presence of a weak magnetic field, atomic hydrogen undergoes the transition:

$$^2P_{1/2} \rightarrow {}^2S_{1/2}\text{, by emission of radiation}$$

Q8. The number of distinct spectral lines that are observed in the resultant Zeeman spectrum is

(a) 2 (b) 3 (c) 4 (d) 6

Ans: (c)

Solution: $^2p_{1/2} \rightarrow {}^2S_{1/2}$ is sodium D1 lines and it has total 4 zeeman components.

Q9. The spectral line corresponding to the transition

$$^2P_{1/2}\left(m_j = +\frac{1}{2}\right) \rightarrow {}^2S_{1/2}\left(m_j = -\frac{1}{2}\right)$$

is observed along the direction of the applied magnetic field. The emitted electromagnetic field is

(a) Circularly polarized (b) Linearly polarized

(c) Unpolarized (d) Not emitted along the magnetic field direction

Ans: (a)

Solution: For $^2P_{1/2}\left(m_j = +\frac{1}{2}\right) \rightarrow {}^2S_{1/2}\left(m_j = -\frac{1}{2}\right)$

Here $\Delta m_j = +1$ gives σ^+ component.

In longitudinal observation σ^+ is circularly polarized.

GATE-2011

Q10. The population inversion in a two layer material **CANNOT** be achieved by optical pumping because

(a) the rate of upward transitions is equal to the rate of downward transitions

(b) the upward transitions are forbidden but downward transitions are allowed

(c) the upward transitions are allowed but downward transitions are forbidden

(d) the spontaneous decay rate of the higher level is very low

Ans: (a)

Q11. A heavy symmetrical top is rotating about its own axis of symmetry (the z-axis). If I_1, I_2 and I_3 are the principal moments of inertia along x, y and z axes respectively, then

(a) $I_2 = I_3$; $I_1 \neq I_2$ (b) $I_1 = I_3$; $I_1 \neq I_2$ (c) $I_1 = I_2$; $I_1 \neq I_3$ (d) $I_1 \neq I_2 \neq I_3$

Ans: (c)

Q12. A neutron passing through a detector is detected because of

(a) the ionization it produces (b) the scintillation light it produces

(c) the electron-hole pairs it produces

(d) the secondary particles produced in a nuclear reaction in the detector medium

Ans: (b)

Q13. An atom with one outer electron having orbital angular momentum l is placed in a weak magnetic field. The number of energy levels into which the higher total angular momentum state splits, is

(a) $2l + 2$ (b) $2l + 1$ (c) $2l$ (d) $2l - 1$

Ans: (b)

Q14. For a multi-electron atom l, L and S specify the one-electron orbital angular momentum, total orbital angular momentum and total spin angular momentum, respectively. The selection rules for electric dipole transition between the two electronic energy levels, specified by l, L and S are

(a) $\Delta L = 0, \pm 1$; $\Delta S = 0$; $\Delta l = 0, \pm 1$ (b) $\Delta L = 0, \pm 1$; $\Delta S = 0$; $\Delta l = \pm 1$

(c) $\Delta L = 0, \pm 1$; $\Delta S = \pm 1$; $\Delta l = 0, \pm 1$ (d) $\Delta L = 0, \pm 1$; $\Delta S = \pm 1$; $\Delta l = \pm 1$

Ans: (b)

Q15. The lifetime of an atomic state is 1 nanosecond. The natural line width of the spectral line in the emission spectrum of this state is of the order of

(a) 10^{-10} eV (b) 10^{-9} eV (c) 10^{-6} eV (d) 10^{-4} eV

Ans: (c)

Solution: $\Delta E = h\Delta\nu = \dfrac{h}{\Delta t} = \dfrac{6.625\times10^{-34}\,J-S}{10^{-9}} = \dfrac{6.625\times10^{-25}}{1.6\times10^{-19}}eV = 4.14\times10^{-6}\,eV$

Q16. The degeneracy of an excited state of nitrogen atom having electronic configuration $1s^2 2s^2 2p^2 3d^1$ is

(a) 6 (b) 10 (c) 15 (d) 150

Ans: (b)

Solution: Degeneracy = 2 (2l + 1)

Q17. The far infrared rotational absorption spectrum of a diatomic molecule shows equilibrium lines with spacing 20 cm^{-1}. The position of the first Stokes line in the rotational Raman spectrum of this molecule is

(a) 20 cm^{-1} (b) 40 cm^{-1} (c) 60 cm^{-1} (d) 120 cm^{-1}

Ans: (c)

Solution: Given $2B = 20$ cm^{-1} $\Rightarrow$ $B = 10$ cm^{-1}

The position of the first stokes line in the rotational Raman spectrum = 6B

$\therefore\ \Delta\bar{\nu} = 6B = 6\times 10 = 60\,cm^{-1}$.

GATE-2012

Q18. The ground state of sodium atom (^{11}Na) is a $^{2}S_{1/2}$ state. The difference in energy levels arising in the presence of a weak external magnetic field B, given in terms of Bohr magnet on, μ_B, is

(a) $\mu_B B$ (b) $2\mu_B B$ (c) $4\mu_B B$ (d) $6\mu_B B$

Ans: (b)

Solution: The energy separation in the Zeeman level is $\Delta E = gM_J(\mu_B B)$

For $^{2}S_{1/2}$ state; $g = 2$ and $M_J = \pm\frac{1}{2}$. Therefore $\Delta E_1 = +(\mu_B B)$ and $\Delta E_2 = -(\mu_B B)$.

Thus $\Delta E = 2\mu_B B$

Q19. The first Stokes line of a rotational Raman spectrum is observed at 12.96 cm^{-1}. Considering the rigid rotor approximation, the rotational constant is given by

(a) 6.48 cm^{-1} (b) 3.24 cm^{-1} (c) 2.16 cm^{-1} (d) 1.62 cm^{-1}

Ans: (c)

Solution: The first Stoke line of the Rotational Raman spectrum lies at = 6B

Thus $6B = 12.96\,cm^{-1} \Rightarrow B = 2.16\ cm^{-1}$.

Q20. Match the typical spectroscopic regions specified in **Group I** with the corresponding type of transitions in **Group II**.

Group I	**Group II**
(P) Infra-red region	(i) electronic transitions involving valence electrons
(Q) Ultraviolet-visible region	(ii) nuclear transitions
(R) X-ray region	(iii) vibrational transitions of molecules
(S) γ-ray region	(iv) transitions involving inner shell electrons

(a) (P, i); (Q, iii); (R, ii); (S, iv) (b) (P, ii); (Q, iv); (R, i); (S, iii)

(c) (P, iii); (Q, i); (R, iv); (S, ii) (d) (P, iv); (Q, i); (R, ii); (S, iii)

Ans: (c)

Q21. The term $\{j_1, j_2\}_J$ arising from $2s^1 3d^1$ electronic in *j-j* coupling scheme are

(a) $\left\{\frac{1}{2},\frac{3}{2}\right\}_{2,1}$ and $\left\{\frac{1}{2},\frac{5}{2}\right\}_{3,2}$ (b) $\left\{\frac{1}{2},\frac{1}{2}\right\}_{1,0}$ and $\left\{\frac{1}{2},\frac{3}{2}\right\}_{2,1}$

(c) $\left\{\frac{1}{2},\frac{1}{2}\right\}_{1,0}$ and $\left\{\frac{1}{2},\frac{5}{2}\right\}_{3,2}$ (d) $\left\{\frac{3}{2},\frac{1}{2}\right\}_{2,1}$ and $\left\{\frac{1}{2},\frac{5}{2}\right\}_{3,2}$

Ans: (a)

Q22. The equilibrium vibration frequency for an oscillator is observed at 2990 cm^{-1}. The ratio of the frequencies corresponding to the first and the fundamental spectral lines is 1.96. Considering the oscillator to be anharmonic, the anharmonicity constant is

(a) 0.005 (b) 0.02 (c) 0.05 (d) 0.1

Ans: (b)

Solution: $\omega_e(1-2x_e) = 2990 cm^{-1}$ and $\frac{2\omega_e(1-3x_e)}{\omega_e(1-2x_e)} = 1.96 \Rightarrow \frac{(1-3x_e)}{(1-2x_e)} = 0.98 \Rightarrow x_e = 0.02$.

GATE-2013

Q23. The number of spectral lines allowed in the spectrum for the $3^2D \rightarrow 3^2P$ transition in sodium is _____.

Ans: 28

Solution: The numbers of Zeeman components for $^2D_{5/2} \rightarrow {}^2P_{3/2}$ transition = 12

The numbers of Zeeman components for $^2D_{3/2} \rightarrow {}^2P_{3/2}$ transition = 10

The numbers of Zeeman components for $^2D_{3/2} \rightarrow {}^2P_{1/2}$ transition = 6

Q24. In a normal Zeeman Effect experiment, spectral splitting of the line at the wavelength 643.8 nm corresponding to the transition $5\,^1D_2 \rightarrow 5\,^1P_1$ of cadmium atoms is to be observed. The spectrometer has a resolution of 0.01 nm. Minimum magnetic field needed to observe this is $\left(m_e = 9.1\times10^{-31}\,kg, e = 1.6\times^{-19}\,C, c = 3\times10^8\,m/s\right)$

(a) $0.26T$ (b) $0.52T$ (c) $2.6T$ (d) $5.2T$

Ans: (b)

Solution: Separation of Zeeman Components

$$\Delta\nu = \frac{eB}{4\pi m} \Rightarrow \Delta\lambda = \frac{\lambda^2}{c}\Delta\nu = \frac{\lambda^2}{c}\frac{eB}{4\pi m}$$

$$B = \frac{4\pi mc}{e}\frac{\Delta\lambda}{\lambda^2} = \frac{4\times3.14\times9.1\times10^{-31}\times3\times10^8}{1.6\times10^{-19}}\times\frac{0.01\times10^{-9}}{\left(643.8\times10^{-9}\right)^2} = 0.514T$$

Q25. The spacing between vibrational energy levels in CO molecule is found to be $8.44\times10^{-2}\,eV$. Given that the reduced mass of CO is $1.14\times10^{-26}\,kg$, Planck's constant is $6.626\times10^{-34}\,Js$ and $1eV = 1.6\times10^{-19}\,J$. The force constant of the bond in CO molecule is

(a) 1.87 N/m (b) 18.7 N/m (c) 187 N/m (d) 1870 N/m

Ans: (c)

Solution: The energy of the quantum harmonic oscillator is

$$E = h\nu\left(n+\frac{1}{2}\right), \qquad n = 0,1,2,.....$$

The frequency of oscillation is $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$.

Where k = Spring constant and μ = reduced mass

The energy levels are equally spaced with energy separation of $\Delta E = h\nu = \frac{h}{2\pi}\sqrt{\frac{k}{\mu}}$

$$k = \left(\frac{2\pi}{h}\Delta E\right)^2\mu = \left(\frac{2\times3.14}{6.626\times10^{-34}}\times8.44\times10^{-2}\times1.6\times10^{-19}\right)^2\times1.14\times10^{-26} = 186.7\,N/m$$

GATE-2014

Q26. The number of normal Zeeman splitting components of $^1P \rightarrow {}^1D$ transition is

(a) 3 (b) 4 (c) 8 (d) 9

Ans: (a)

Solution: This is singlet transition.

Q27. The moment of inertia of a rigid diatomic molecule A is 6 times that of another rigid diatomic molecule B. If the rotational energies of the two molecules are equal, then the corresponding values of the rotational quantum numbers J_A and J_B are

(a) $J_A = 2, J_B = 1$ (b) $J_A = 3, J_B = 1$

(c) $J_A = 5, J_B = 0$ (d) $J_A = 6, J_B = 1$

Ans: (b)

Solution: $\frac{J_A(J_A+1)}{J_B(J_B+1)} = \frac{I_B}{I_A} = \frac{I_B}{6I_B} \Rightarrow J_A = 6,\ J_B = 1$

Q28. The value of the magnetic field required to maintain non-relativistic protons of energy $1MeV$ in a circular orbit of radius 100mm is_____Tesla

(Given: $m_p = 1.67 \times 10^{-27} kg,\ \ e = 1.6 \times 10^{-19} C$)

Ans: 1.44

Solution: $\frac{mv^2}{r} = qvB,\ E = \frac{1}{2}mv^2 \Rightarrow B = \frac{\sqrt{2mE}}{qr} = 1.44$

Q29. Neutrons moving with speed $10^3\ m/s$ are used for the determination of crystal structure. If the Bragg angle for the first order diffraction is 30^o the interplannar spacing of the crystal is _____ $\overset{0}{A}$. (Given: $m_n = 1.675 \times 10^{-27} kg\ \ h = 6.626 \times 10^{-34} J.s$)

Ans: 4

Solution: $2d \sin\theta = \lambda = \frac{h}{mv} \Rightarrow 2d \sin 30^0 = \frac{6.62 \times 10^{-34}}{1.67 \times 10^{-27} \times 10^3} \Rightarrow d = 4\ \overset{0}{\text{A}}$

Q30. The emission wavelength for the transition $D_2 \rightarrow F_3$ is 3122 Å. The ratio of population of the final to the initial states at a temperature $5000K$ is $\left(h = 6.626 \times 10^{-34} J\text{-}s,\ \ c = 3 \times 10^8 m/s,\ \ k_B = 1.380 \times 10^{-23} J/K\right)$

(a) 2.03×10^{-5} (b) 4.02×10^{-5} (c) 7.02×10^{-5} (d) 9.83×10^{-5}

Ans: (c)

Solution: $\frac{N_F}{N_I}=\frac{(2J_f+1)}{(2J_i+1)}e^{-\frac{hc}{\lambda k_B T}}=\frac{5}{7}e^{-9.227641144}=7.02\times10^{-5}$

GATE-2015

Q31. In a rigid rotator of mass M, if the energy of the first excited state is ($1\ meV$), then the fourth excited state energy (in meV) is _____.

Ans.: 10

Solution: $\because E\propto J(J+1)$ where $J=0,1,2,3..$

$\Rightarrow\frac{E_4}{E_1}=\frac{4(4+1)}{1(1+1)}\Rightarrow E_4=10E_1=10\,meV$, where $J=0,1,2,3..$

Q32. The binding energy per molecule of $NaCl$ (lattice parameter is $0.563\,nm$) is $7.956\ eV$. The repulsive term of the potential is of the form $\frac{K}{r^9}$, where K is a constant. The value of the Modelung constant is _____ (upto three decimal places)

(Electron charge $e=-1.6\times10^{-19}C$; $\varepsilon_0=8.854\times10^{-12}\,C^2N^{-1}m^{-2}$)

Ans.: 2.80

Solution: The total energy of one ion due to the presence of all others in NaCl crystal is (considering univalent ions)

$U(r)=-\frac{Ae^2}{4\pi\varepsilon_0 r}+\frac{K}{r^n}$, where A is Modelung Constant.

The potential energy will be minimum at the equilibrium spacing r_0.

Thus $\left[\frac{dU}{dr}\right]_{r=r_0}=\left[\frac{Ae^2}{4\pi\varepsilon_0 r_0^2}-\frac{Kn}{r_0^{n+1}}\right]=0\Rightarrow K=\frac{Ae^2r_0^{n-1}}{4\pi\varepsilon_0 n}$

Thus, Binding energy of molecule or lattice energy is

$$U_0=[U]_{r=r_0}=-\frac{Ae^2}{4\pi\varepsilon_0 r_0}+\frac{Ae^2r_0^{n-1}}{4\pi\varepsilon_0 nr_0^n}=-\left[\frac{Ae^2}{4\pi\varepsilon_0 r_0}\right]\left[\frac{n-1}{n}\right]$$

Given repulsive term of the potential is $\frac{K}{r^9}(\therefore n=9)$

Also binding energy per molecule is $U_0=7.95\ eV$

The Modelung constant is $A = U_0 \times \frac{4\pi\varepsilon_0 r_0}{e^2} \times \frac{n}{n-1}$ and the lattice parameter is $a = 0.563\ nm$. Thus the interatomic separation is $r_0 = \frac{a}{2} = 0.28\ nm$.

$$\Rightarrow A = 7.95 \times 1.67 \times 10^{-19}\ J \times \frac{4 \times 3.14 \times \left(8.85 \times 10^{-12} C^2 N^{-1} m^{-2}\right) \times \left(0.282 \times 10^{-9}\right)}{\left(1.67 \times 10^{-19} J\right)^2} \times \frac{9}{8}$$

$$\Rightarrow A = \frac{7.95 \times 1.67 \times 4 \times 3.14 \times 8.85 \times 0.282 \times 9}{1.67 \times 8} \times 10^{-2} \Rightarrow A = 2.80$$

Q33. Match the phrases in Group I and Group II and identify the correct option.

Group I	Group II
(P) Electron spin resonance (ESR)	(i) radio frequency
(Q) Nuclear magnetic resonance (NMR)	(ii) visible range frequency
(R) Transition between vibrational states of a molecule	(iii) microwave frequency
(S) Electronic transition	(iv) far-infrared range

(a) (P-i), (Q-ii), (R-iii), (S-iv) (b) (P-ii), (Q-i), (R-iv), (S-iii)

(c) (P-iii), (Q-iv), (R-i), (S-ii) (d) (P-iii), (Q-i), (R-iv), (S-ii)

Ans.: (d)

Solution: (P) Electron spin resonance (ESR) is achieved by Microwave frequency (iii)

(Q): Nuclear magnetic resonance (NMR) is achieved by Radio frequency (i)

(R): Transition between vibrational states of a molecule is achieved by radiation of far infrared range (iv)

(S): Electronic transition is achieved by visible radiation (ii)

Q34. The excitation wavelength of laser in a Raman effect experiment is $546\,nm$. If the Stokes' line is observed at $552\,nm$, then the wavenumber of the anti-Stokes' line (in cm^{-1}) is _____

Ans.: 18514

Solution: Raman displacement is

$$\Delta\bar{\nu} = \bar{\nu}_{AS} - \bar{\nu}_0 = \bar{\nu}_0 - \bar{\nu}_S \text{ or } \Delta\bar{\nu} = \frac{1}{\lambda_{AS}} - \frac{1}{\lambda_0} = \frac{1}{\lambda_0} - \frac{1}{\lambda_S}$$

where λ_{AS}, λ_0, λ_S are wavelength of anti-stoke, exciting & stoke line.

From above relation we can write

$$\frac{1}{\lambda_{AS}}-\frac{1}{\lambda_0}=\frac{1}{\lambda_0}-\frac{1}{\lambda_S}\Rightarrow\frac{1}{\lambda_{AS}}=\frac{2}{\lambda_0}-\frac{1}{\lambda_S}\Rightarrow\frac{1}{\lambda_{AS}}=\frac{2\lambda_S-\lambda_0}{\lambda_0\lambda_S}\Rightarrow\lambda_{AS}=\frac{\lambda_0\lambda_S}{2\lambda_S-\lambda_0}$$

$$\therefore\lambda_{AS}=\frac{\left(546\times10^{-9}m\right)\left(552\times10^{-9}m\right)}{\left(2\times552\times10^{-9}m-546\times10^{-9}m\right)}=\frac{546\times552}{558}\times10^{-9}m$$

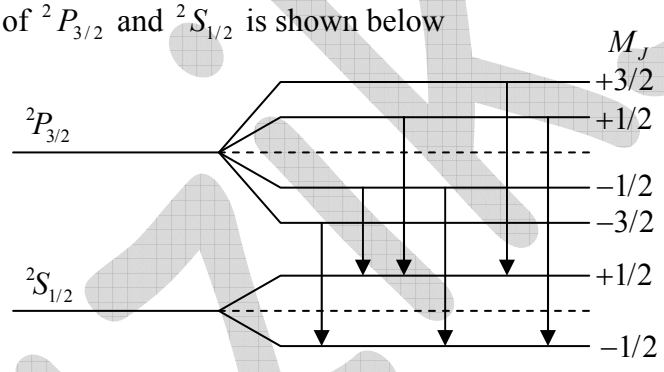
$$\Rightarrow\lambda_{AS}=540.129\times10^{-9}m=540.129\times10^{-7}cm$$

Anti-stoke wavenumber is $\bar{\nu}_{AS}=\frac{1}{\lambda_{AS}}=\frac{1}{540.129\times10^{-7}cm}=18514\ \ cm^{-1}$

Q35. The number of permitted transitions from $^2P_{3/2}\rightarrow{}^2S_{1/2}$ in the presence of a weak magnetic field is _____

Ans. : 6

Solution: Zeeman splitting of $^2P_{3/2}$ and $^2S_{1/2}$ is shown below



The selection rule for Zeeman transactions are

$$\Delta M_J=0,\ \pm1\left(0\not\rightarrow0\ \text{ if }\ J=0\right)$$

There are total six transition in accordance with above selection rules.

GATE-2016

Q36. The molecule $^{17}O_2$

(a) Raman active but not NMR (nuclear magnetic resonance) active.

(b) Infrared active and Raman active but not NMR active.

(c) Raman active and NMR active.

(d) Only NMR active.

Ans.: (c)

Solution: (i) Molecule $^{17}O_2$ can not absorb infrared as there is no change in dipole moment during vibration. Thus $^{17}O_2$ is infrared inactive.

(ii) Molecule $^{17}O_2$ shows change in polaraziability during rotation. Thus it is Raman active molecule.

(iii) The nucleus of ^{17}O has spin $\frac{5}{2}$, therefore it is NMR active.

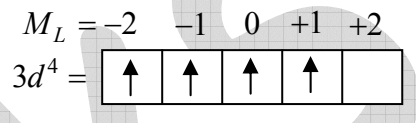
Q37. There are four electrons in the $3d$ shell of an isolated atom. The total magnetic moment of the atom in units of Bohr magneton is _____.

Ans.: 0

Solution: The configuration leads to $S=2$ and $L=2$

Since it is the case of less than half filled sub shell, thus according to Hund's rules, lower J will be in ground state.

$\therefore J = L - S = 0 \;\; \vec{\mu}_J = -g\left(\frac{e}{2m}\right)\vec{J}$.



Thus, $\vec{\mu}=0$

Q38. Which of the following transitions is NOT allowed in the case of an atom, according to the electric dipole radiation selection rule?

(a) $2s-1s$ (b) $2p-1s$ (c) $2p-2s$ (d) $3d-2p$

Ans.: (a)

Solution: In electron dipole transition, $\Delta l = \pm 1$. Thus in transition $2s \to 1s$, $\Delta l = 0$. It violate the selection rule and hence not allowed.

Q39. The number of spectroscopic terms resulting from the $L.S$ coupling of a $3p$ electron and a $3d$ electron is _____.

Ans.: 12

Solution: For $3p^1 3d^1$: $s_1 = \frac{1}{2}, s_2 = \frac{1}{2} \;\; \therefore S = 0,1$

$l_1 = 1, l_2 = 2 \quad \therefore L = 1,2,3$

$S=0, L=1 \Rightarrow J=1 \quad \therefore \text{Term} = {}^1P_1$

$S=0, L=2 \Rightarrow J=2 \quad \therefore \text{Term} = {}^1D_2$

$S=0, L=3 \Rightarrow J=3 \quad \therefore \text{Term} = {}^1F_3$

$S=1, L=1 \Rightarrow J=0,1,2 \quad \therefore \text{Terms} = {}^3P_0, {}^3P_1, {}^3P_2$

$S=1, L=2 \Rightarrow J=1,2,3 \qquad \therefore$ Terms $= {}^3D_1, {}^3D_2, {}^3D_3$

$S=1, L=3 \Rightarrow J=2,3,4 \qquad \therefore$ Terms $= {}^3F_2, {}^3F_3, {}^3F_4$

Thus total number of spectroscopic terms are 12.

GATE-2017

Q40. The wavefunction of which orbital is spherically symmetric:

(a) p_x (b) p_y (c) s (d) d_{xy}

Ans. : (c)

Solution: For s orbital $l=0$

Q41. The total energy of an inert-gas crystal is given by $E(R)=\frac{0.5}{R^{12}}-\frac{1}{R^6}$ (in eV), where R is the inter-atomic spacing in Angstroms. The equilibrium separation between the atoms is Angstroms. (up to two decimal places)

Ans. : 1

Solution: Given that $E(R)=\frac{0.5}{R^{12}}-\frac{1}{R^6}$

For equilibrium separation

$$\frac{dE}{dR}=0 \Rightarrow \frac{dE}{dR}=-\frac{12\times 0.5}{R^{13}}+\frac{6}{R^7}=0 \Rightarrow \frac{1}{R^6}\left[\frac{-6}{R^6}+6\right]=0 \Rightarrow R=1$$

Q42. Which one of the following gases of diatomic molecules is Raman, infrared, and NMR active?

(a) ${}^1H$-${}^1H$ (b) ${}^{12}C$-${}^{16}O$ (c) ${}^1H$-${}^{35}Cl$ (d) ${}^{16}O$-${}^{16}O$

Ans. : (c)

Solution:

(a) ${}^1H-{}^1H$ Infrared inactive

(b) ${}^{12}C-{}^{16}O$ NMR Inactive

(c) ${}^1H-{}^{35}Cl$ Raman, infrared & NMR active

(d) ${}^{16}O-{}^{16}O$ Infrared , Raman inactive

Q43. Using Hund's rule the total angular momentum quantum number J for the electronic ground state of the nitrogen atom is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 0 (d) 1

Ans. : (b)

Solution: $N:7:1s^2 2s^2 2p^3$

For p^3 : $M_L =$

-1	0	$+1$
↑	↑	↑

$\therefore$ spectral term $= {}^{2s+1}L_J = {}^4s_{3/2}$

Q44. Positronium is an atom made of an electron and a positron. Given the Bohr radius for the ground state of the Hydrogen atom to be 0.53 Angstroms, the Bohr radius for the ground state of positronium is……Angstroms. (up to two decimal places).

Ans. : 1.06

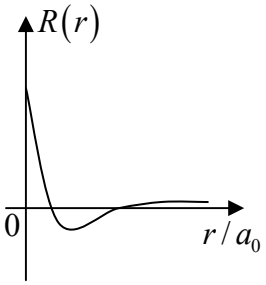
Solution: $r_n = a_0\left(\frac{m_e}{\mu}\right)$

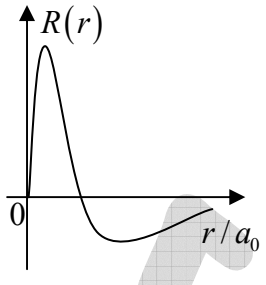
When $\mu = \frac{m_e m_e}{m_e + m_e} = \frac{m_e^2}{2m_e} = \frac{m_e}{2}$

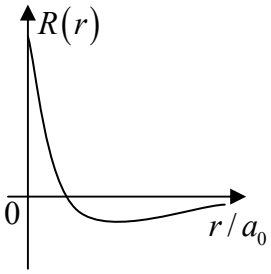
$\therefore \quad r_n = 2a_0 = 2\times 0.53 = 1.06 A^0$

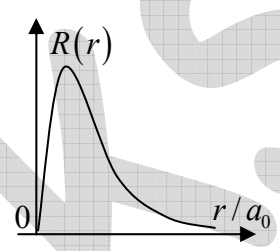
GATE-2018

Q45. Which one of the following represents the $3p$ radial wave function of hydrogen atom? (a_0 is the Bohr radius)

(a) 

(b) 

(c) 

(d) 

Ans. : (b)

Solution: $3p$ radial wave function is $R_{31} \alpha\ r\left(1-\frac{r}{6a_0}\right)e^{-\frac{r}{3a_0}}$

Q46. Given the following table,

Group I	Group II
P: Stern-Gerlach experiment	1: Wave nature of particles
Q: Zeeman effect	2: Quantization of energy of electrons in the atoms
R: Frank-Hertz experiment	3: Existence of electron spin
S: Davisson-Germer experiment	4: Space quantization of angular momentum

Which one of the following correctly matches the experiments from Group I to their inferences in Group II?

(a) P-2, Q-3, R-4, S-1 (b) P-1, Q-3, R-2, S-4

(c) P-3, Q-4, R-2, S-1 (d) P-2, Q-1, R-4, S-3

Ans. : (c)

Q47. The expression for the second overtone frequency in the vibrational absorption spectra of a diatomic molecule in terms of the harmonic frequency ω_e and anharmonicity constant x_e is

(a) $2\omega_e(1-x_e)$ (b) $2\omega_e(1-3x_e)$ (c) $3\omega_e(1-2x_e)$ (d) $3\omega_e(1-4x_e)$

Ans. : (d)

Solution: $\varepsilon_V = \omega_e\left(v+\frac{1}{2}\right)-\omega_e x_e\left(v+\frac{1}{2}\right)^2$

Second overtone $v=0 \rightarrow v=3$

$$\therefore\ \bar{v} = \varepsilon_{v=3}-\varepsilon_{v=0} = \frac{7}{2}\omega_e-\omega_e x_e\left(\frac{7}{2}\right)^2-\frac{\omega_e}{2}+\omega_e x_e\left(\frac{1}{2}\right)^2 = 3\omega_e-12\omega_e x_e = 3w_e(1-4x_e)$$

Q48. Match the physical effects and order of magnitude of their energy scales given below, where $\alpha = \frac{e^2}{4\pi \in_0 \hbar c}$ is fine structure constant; m_e and m_p are electron and proton mass, respectively.

Group I	Group II
P: Lamb shift	1: $\sim O\left(\alpha^2 m_e c^2\right)$
Q: Fine structure	2: $\sim O\left(\alpha^4 m_e c^2\right)$
R: Bohr energy	3: $\sim O\left(\alpha^4 m_e^2 c^2 / m_p\right)$
S: Hyperfine structure	4: $\sim O\left(\alpha^5 m_e c^2\right)$

(a) P-3, Q-1, R-2, S-4 (b) P-2, Q-3, R-1, S-4

(c) P-4, Q-2, R-1, S-3 (d) P-2, Q-4, R-1, S-3

Ans. : (c)

Solution:- Bohr energy $\Delta E \propto \alpha^2 m_e c^2$

Fine structure $\Delta E \propto \alpha^4 m_e c^2$

Lamb straight $\Delta E \propto \alpha^5 m_e c^2$

Hyperfine structure $\Delta E \propto \frac{\alpha^4 m_e c^2}{m_p}$

Q49. The intrinsic/permanent electric dipole moment in the ground state of hydrogen atom is (a_0 is the Bohr radius)

(a) $-3ea_0$ (b) zero (c) ea_0 (d) $3ea_0$

Ans. : (b)

Solution: For dipole moment energy is $-eEr\cos\theta$

$$E_1^1 = \langle -eEr\cos\theta \rangle = eE\langle r\rangle\langle\cos\theta\rangle = 0 \qquad [\because \langle\cos\theta\rangle = 0]$$

Q50. Which one of the following is an allowed electric dipole transition?

(a) $^1S_0 \to {}^3S_1$ (b) $^2P_{3/2} \to {}^2D_{5/2}$ (c) $^2D_{5/2} \to {}^2P_{1/2}$ (d) $^3P_0 \to {}^5D_0$

Ans. : (b)

Solution: For electric dipole transition

$\Delta L = 0, \pm 1 \quad (0 \not\to 0)$, $\Delta J = 0. \pm 1$, $\Delta S = 0$

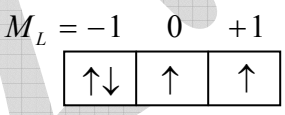
Only option (b) satisfies above selection rules

Q51. The term symbol for the electronic ground state of oxygen atom is

(a) $^1S_0$ (b) $^1D_2$ (c) $^3P_0$ (d) $^3P_2$

Ans. : (d)

Solution: O: $1s^2, 2s^2, 2p^4$



Here, $S = 1$, $L = 2$

According to Hund's rule, for ground state energy

$J = (L+S) = 2 \qquad \therefore \quad {}^{2S+1}L_J = {}^3P_2$

Q52. $4\,MeV$ γ- rays emitted by the de-excitation of ^{19}F are attributed, assuming spherical symmetry, to the transition of protons from $1d_{3/2}$ state to $1d_{5/2}$ state. If the contribution of spin-orbit term to the total energy is written as $C\langle \vec{l}\cdot\vec{s}\rangle$, the magnitude of C is _____ MeV (up to one decimal place).

Ans. : 1.6

Solution: $l = 1, s = \frac{1}{2}, \hat{j}_1 = \frac{3}{2}, \hat{j}_2 = \frac{5}{2}$

$$\vec{j} = (\vec{l} + \vec{s}) \Rightarrow j^2 = l^2 + s^2 + 2\vec{l}\cdot\vec{s} \Rightarrow \vec{l}\cdot\vec{s} = \frac{(j^2 + l^2 - s^2)}{2}$$

$$\left\langle \vec{l}\cdot\vec{s}\right\rangle = \frac{\left[j(j+1)-(l+1)-s(s+1)\right]\hbar^2}{2}$$

$$\Delta E = \alpha\left[\left\langle \vec{l}\cdot\vec{s}\right\rangle_{5/2} - \left\langle \vec{l}\cdot\vec{s}\right\rangle_{3/2}\right] = \alpha\left[\frac{5}{2}\cdot\frac{7}{2} - \frac{3}{2}\cdot\frac{5}{2}\right]\frac{\hbar^2}{2} = \alpha\cdot\left(\frac{20}{8}\right)\hbar^2 = \frac{20}{8}\cdot C$$

$$\Delta E = \frac{20}{8}C = 4MeV \Rightarrow C = \frac{32}{20}MeV, C = 1.6\,MeV\,.$$

Q53. An atom in its singlet state is subjected to a magnetic field. The Zeeman splitting of its 650 nm spectral line is 0.03 nm. The magnitude of the field is _____ Tesla (up to two decimal places).

$(e = 1.60\times10^{-19}C, m_e = 9.11\times10^{-31}kg, c = 3.0\times10^{8}ms^{-1})$

Ans. : 1.52

Solution: $\Delta\lambda = \frac{\lambda^2}{c}\times\frac{eB}{4\pi m}$

$$\Rightarrow B = \frac{c}{\lambda^2}\cdot\frac{4\pi m}{e}\Delta\lambda = \frac{3\times10^{8}}{\left(650\times10^{-9}\right)^2}\cdot\frac{4\pi\times9.1\times10^{-31}}{1.6\times10^{-19}}\cdot\left(0.03\times10^{-9}\right) = 1.52T$$

GATE-2019

Q54. The spin-orbit interaction term of an electron moving in a central field is written as $f(r)\vec{l}\cdot\vec{s}$, where r is the radial distance of the electron from the origin. If an electron moves inside a uniformly charged sphere, then

(a) $f(r) =$ constant (b) $f(r)\propto r^{-1}$ (c) $f(r)\propto r^{-2}$ (d) $f(r)\propto r^{-3}$

Ans. : (a)

Solution: The electric potential of a uniformly charged sphere at $r < R$ is

$$V = \frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

where Q is the electric charge on the sphere of radius R and k is a constant.

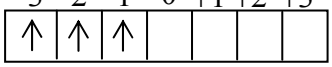
The interaction energy is $W = f(r)\vec{l}\cdot\vec{s}$, where for central potential V, $f(r) = \frac{1}{r}\left(\frac{\partial V}{\partial r}\right)$

$\therefore f(r) = \frac{1}{r}\left[\frac{-kQr}{R^3}\right] = \frac{-kQ}{R^3} =$ constant. Thus option (a) is correct.

Q55. The ground state electronic configuration of the rare-earth ion (Nd^{3+}) is $[Pd]4f^3 5s^2 5p^6$. Assuming LS coupling, the Lande g - factor of this ion is $\frac{8}{11}$. The effective magnetic moment in units of Bohr magneton μ_B (rounded off to two decimal places) is _____

Ans.: 3.62

Solution: For $4f^3$ $M_L =$ $-3\ -2\ -1\ \ 0\ +1\ +2\ +3$ $L=6,\quad S=3/2,\quad J=9/2$



$$\therefore \mu = g_J \mu_B \sqrt{J(J+1)} = \frac{8}{11} \times \mu_B \times \sqrt{\frac{9}{2}\left(\frac{9}{2}+1\right)}$$

$$= \frac{8}{11}\sqrt{\frac{9}{2}\times\frac{11}{2}}\mu_B = 3.62\,\mu_B$$

SOLID STATE PHYSICS SOLUTIONS

GATE-2010

Q1. The valence electrons do not directly determine the following property of a metal

(a) Electrical conductivity (b) Thermal conductivity

(c) Shear modulus (d) Metallic luster

Ans: (c)

Q2. Consider *X*-ray diffraction from a crystal with a face-centered-cubic (*fcc*) lattice. The lattice plane for which there is NO diffraction peak is

(a) (2, 1, 2) (b) (1, 1, 1) (c) (2, 0, 0) (d) (3, 1, 1)

Ans: (a)

Q3. The Hall coefficient, R_H, of sodium depends on

(a) The effective charge carrier mass and carrier density

(b) The charge carrier density and relaxation time

(c) The charge carrier density only

(d) The effective charge carrier mass

Ans: (c)

Q4. The Bloch theorem states that within a crystal, the wavefunction, $\psi(\vec{r})$, of an electron has the form

(a) $\psi(\vec{r}) = u(\vec{r})e^{i\vec{k}.\vec{r}}$ where u($\vec{r}$) is an arbitrary function and $\vec{k}$ is an arbitrary vector

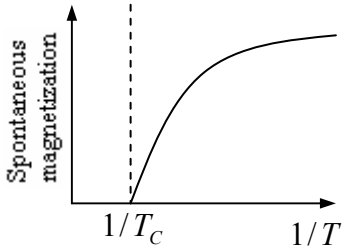
(b) $\psi(\vec{r}) = u(\vec{r})e^{i\vec{G}.\vec{r}}$ where u($\vec{r}$) is an arbitrary function and $\vec{G}$ is a reciprocal lattice vector

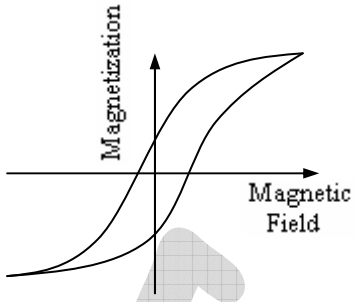
(c) $\psi(\vec{r}) = u(\vec{r})e^{i\vec{G}.\vec{r}}$ where $u(\vec{r}) = u(\vec{r} + \vec{\Lambda})$, $\vec{\Lambda}$ is a lattice vector and $\vec{G}$ is a reciprocal lattice vector

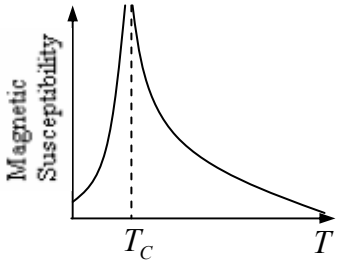
(d) $\psi(\vec{r}) = u(\vec{r})e^{i\vec{k}.\vec{r}}$ where $u(\vec{r}) = u(\vec{r} + \vec{\Lambda})$, $\vec{\Lambda}$ is a lattice vector and $\vec{k}$ is an arbitrary vector

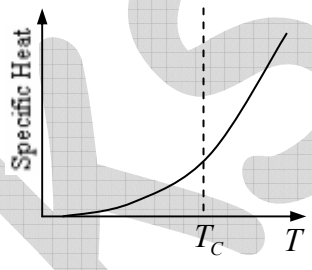
Ans: (d)

Q5. In an experiment involving a ferromagnetic medium, the following observations were made. Which one of the plots does NOT correctly represent the property of the medium? (T_C is the Curie temperature)

(a)


(b)


(c)


(d)


Ans: (c)

Q6. The thermal conductivity of a given material reduces when it undergoes a transition from its normal state to the superconducting state. The reason is:

(a) The Cooper pairs cannot transfer energy to the lattice

(b) Upon the formation of Cooper pairs, the lattice becomes less efficient in heat transfer

(c) The electrons in the normal state lose their ability to transfer heat because of their coupling to the Cooper pairs

(d) The heat capacity increases on transition to the superconducting state leading to a reduction in thermal conductivity

Ans: (d)

Q7. For a two-dimensional free electron gas, the electronic density n, and the Fermi energy E_F, are related by

(a) $n = \dfrac{(2mE_F)^{3/2}}{3\pi^2\hbar^3}$ (b) $n = \dfrac{mE_F}{\pi\hbar^2}$ (c) $n = \dfrac{mE_F}{2\pi\hbar^2}$ (d) $n = \dfrac{2^{1/3}(mE_F)^{1/3}}{\pi\hbar}$

Ans: (b)

Solution: For two dimensional gas, the number of possible k-states between k and $k+dk$ is

$g(k)dk = \left(\frac{L}{2\pi}\right)^2 2\pi k\, dk = 2\left(\frac{L}{2\pi}\right)^2 2\pi k\, dk$ it is multiplied by 2 for electron gas

Since $k^2 = \frac{2mE}{\hbar^2} \because 2k\, dk = \frac{2m}{\hbar^2} dE \Rightarrow 2\pi k\, dk = \frac{2\pi m}{\hbar^2} dE$

$\therefore\ g(E)dE = 2\left(\frac{L}{2\pi}\right)^2 \cdot \frac{2\pi m}{\hbar^2} dE$

The total number of electrons at $T = 0^0 K$ is

$$N = \int_0^{E_F} g(E)dE \times F(E) = \int_0^{E_F} g(E)dE = 2\pi \cdot \frac{2m}{\hbar^2}\left(\frac{L}{2\pi}\right)^2 \int_0^{E_F} dE = 2\pi \cdot \frac{2m}{\hbar^2} \cdot \frac{L^2}{4\pi^2} \cdot E_F$$

$$\Rightarrow N = \frac{m}{\pi\hbar^2} \cdot L^2 E_F \Rightarrow E_F = \frac{\pi\hbar^2}{m}\left(\frac{N}{L^2}\right) = \frac{\pi\hbar^2}{m} \cdot n \Rightarrow n = \frac{mE_F}{\pi\hbar^2}$$

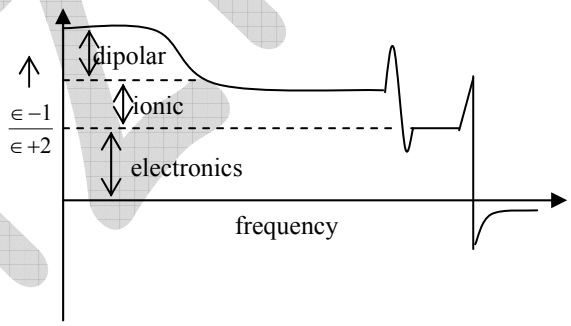
Q8. Far away from any of the resonance frequencies of a medium, the real part of the dielectric permittivity is

(a) Always independent of frequency (b) Monotonically decreasing with frequency

(c) Monotonically increasing with frequency (d) A non-monotonic function of frequency

Ans: (a)

Solution:



GATE-2011

Q9. The temperature (T) dependence of magnetic susceptibility (χ) of a ferromagnetic substance with a Curie temperature (T_c) is given by

(a) $\frac{C}{T-T_c}$, for $T < T_c$ (b) $\frac{C}{T-T_c}$, for $T > T_c$

(c) $\frac{C}{T+T_c}$, for $T > T_c$ (d) $\frac{C}{T+T_c}$, for all temperatures

where C is constant .

Ans: (b)

Q10. The order of magnitude of the energy gap of a typical superconductor is

(a) 1 MeV (b) 1 KeV (c) 1 eV (d) 1 meV

Ans: (d)

Q11. For a three-dimensional crystal having N primitive unit cells with a basis of p atoms, the number of optical branches is

(a) 3 (b) $3p$ (c) $3p-3$ (d) $3N-3p$

Ans: (c)

Q12. For an intrinsic semiconductor, m_e^* and m_h^* are respectively the effective masses of electrons and holes near the corresponding band edges. At a finite temperature the position of the Fermi level

(a) depends on m_e^* but not on m_h^* (b) depends on m_h^* but not on m_e^*

(c) depends on both m_e^* and m_h^* (d) depends neither on m_e^* nor on m_h^*

Ans: (c)

Solution: The Fermi level for intrinsic semicondutor is $E_F = \frac{E_c + E_v}{2} + \frac{3}{4}k_B T \ln\left(\frac{m_h^*}{m_e^*}\right)$

Q13. A metal with body centered cubic (bcc) structure show the first (i.e. smallest angle) diffraction peak at a Bragg angle of $\theta = 30^o$. The wavelength of X-ray used is 2.1 Å. The volume of the PRIMITIVE unit cell of the metal is

(a) 26.2 $(Å)^3$ (b) 13.1$(Å)^3$ (c) 9.3 $(Å)^3$ (d) 4.6 $(Å)^3$

Ans: (b)

Solution: According to Bragg's law $2d\sin\theta = \lambda$ where $d = \dfrac{a}{\sqrt{h^2+k^2+l^2}}$

For BCC structure the first diffraction peak appear for (110) plane.

$$\because d = \frac{a}{\sqrt{2}} \Rightarrow \frac{2a}{\sqrt{2}}\sin 30^0 = \lambda \Rightarrow \sqrt{2}a\sin 30^0 = 2.1A^0$$

$$\Rightarrow \sqrt{2}\,a\times\frac{1}{2} = 2.1A^0 \Rightarrow a = \sqrt{2}\times 2.1A^0 \Rightarrow a = 2.97A^0.$$

The volume primitive unit cell of BCC is volume $= \dfrac{a^3}{2} = \dfrac{26.2}{2}\left(A^0\right)^3 = 13.1\left(A^0\right)^3$

Common Data for Questions 14 and 15:

The tight binding energy dispersion (E-k) relation for electrons in a one-dimensional array of atoms having lattice constant a and total length L is

$$E = E_0 - \beta - 2\gamma\, cos\,(ka),$$

where E_0, β and γ are constants and k is the wave vector.

Q14. The density of states of electrons (including spin degeneracy) in the band is given by

(a) $\dfrac{L}{\pi\gamma\, a\sin(ka)}$ (b) $\dfrac{L}{2\pi\gamma\, a\sin(ka)}$ (c) $\dfrac{L}{2\pi\gamma\, a\cos(ka)}$ (d) $\dfrac{L}{\pi\gamma\, a\cos(ka)}$

Ans: (a)

Solution: $D(E) = 2\times 2\left(\dfrac{L}{2\pi}\right)\cdot\dfrac{1}{dE/dk} = 2\left(\dfrac{L}{2\pi}\right)\cdot\dfrac{2\times 1}{2a\gamma\sin(ka)} = \dfrac{2\times L}{2\pi\gamma a\sin(ka)}$

Q15. The effective mass of electrons in the band is given by

(a) $\dfrac{\hbar^2}{\gamma\, a^2\cos(ka)}$ (b) $\dfrac{\hbar^2}{2\gamma\, a^2\cos(ka)}$ (c) $\dfrac{\hbar^2}{\gamma\, a^2\sin(ka)}$ (d) $\dfrac{\hbar^2}{2\gamma\, a^2\sin(ka)}$

Ans: (b)

Solution: Effective mass $m^* = \dfrac{\hbar^2}{\left(\dfrac{d^2E}{dk^2}\right)} = \dfrac{\hbar^2}{2a^2\gamma\cos(ka)} = \dfrac{\hbar^2}{2\gamma a^2\cos(ka)}$

GATE-2012

Q16. For an ideal Fermi gas in three dimensions, the electron velocity V_F at the Fermi surface is related to electron concentration n as,

(a) $V_F \propto n^{2/3}$ (b) $V_F \propto n$ (c) $V_F \propto n^{1/2}$ (d) $V_F \propto n^{1/3}$

Ans: (d)

Solution: $V_F = \frac{\hbar}{m}\left(3\pi^2 n\right)^{1/3}$

Q17. The total energy, E of an ideal non-relativistic Fermi gas in three dimensions is given by $E \propto \frac{N^{5/3}}{V^{2/3}}$ where N is the number of particles and V is the volume of the gas. Identify the **CORRECT** equation of state (P being the pressure),

(a) $PV = \frac{1}{3}E$ (b) $PV = \frac{2}{3}E$ (c) $PV = E$ (d) $PV = \frac{5}{3}E$

Ans: (b)

Q18. Which one of the following CANNOT be explained by considering a harmonic approximation for the lattice vibrations in solids?

(a) Deby's T^3 law (b) Dulong Petit's law

(c) Optical branches in lattices (d) Thermal expansion

Ans: (d)

Solution: Thermal expansion in solid can only be explained if solid behave as a anharmonic oscillator.

Q19. A simple cubic crystal with lattice parameter a_c undergoes transition into a tetragonal structure with lattice parameters $a_t = b_t = \sqrt{2}a_c$ and $c_t = 2a_c$, below a certain temperature. The ratio of the interplanar spacing of (1 0 1) planes for the cubic and the tetragonal structure is

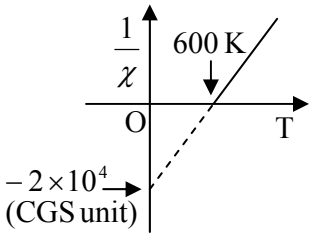
(a) $\sqrt{\frac{1}{6}}$ (b) $\frac{1}{6}$ (c) $\sqrt{\frac{3}{8}}$ (d) $\frac{3}{8}$

Ans: (c)

Solution: For Cubic Lattice $d_c = \frac{a}{\sqrt{h^2+k^2+l^2}} = \frac{a_c}{\sqrt{2}}$

For Tetragonal lattice $d_t = \frac{a}{\sqrt{\frac{h^2+k^2}{a^2}+\frac{l^2}{c^2}}} = \frac{2a_c}{\sqrt{3}}$. Therefore, the ratio is $\frac{d_c}{d_t} = \sqrt{\frac{3}{8}}$

Q20. Inverse susceptibility ($1/\chi$) as a function of temperature, T for a material undergoing paramagnetic to ferromagnetic transition is given in the figure, where O is the origin. The values of the Curie constant, C, and the Weiss molecular field constant, λ, in CGS units, are



(a) $C = 5\times10^{-5}, \lambda = 3\times10^{-2}$ (b) $C = 3\times10^{-2}, \lambda = 5\times10^{-5}$

(c) $C = 3\times10^{-2}, \lambda = 2\times10^{4}$ (d) $C = 2\times10^{4}, \lambda = 3\times10^{-2}$

Ans: (c)

Solution: $\frac{1}{\chi} = \frac{T - T_C}{C}$ and $T_C = \lambda C$. Here $T_C = 600K$ and $\frac{1}{\chi} = -2\times10^{4}$

Thus $C = 3\times10^{-2}$ and $\lambda = 2\times10^{4}$.

Common Data for Questions 21–22

The dispersion relation for a one dimensional monoatomic crystal with lattice spacing a, which interacts nearest neighbour harmonic potential is given by

$$\omega = A\left|\sin\frac{Ka}{2}\right|$$

where A is a constant of appropriate unit.

Q21. The group velocity at the boundary of the first Brillouin zone is

(a) 0 (b) 1 (c) $\sqrt{\frac{Aa^2}{2}}$ (d) $\frac{1}{2}\sqrt{\frac{Aa^2}{2}}$

Ans: (a)

Solution: At the first Brillouin zone the frequency is maximum and the group velocity which is the derivative of the angular frequency is zero.

Q22. The force constant between the nearest neighbour of the lattice is (M is the mass of the atom)

(a) $\frac{MA^2}{4}$ (b) $\frac{MA^2}{2}$ (c) MA^2 (d) $2MA^2$

Ans: (a)

Solution: $\because \omega = \sqrt{\frac{4\beta}{M}}\sin\left(\frac{ka}{2}\right) \Rightarrow A = \sqrt{\frac{4\beta}{M}} \Rightarrow \beta = \frac{MA^2}{4}$

GATE-2013

Q23. A phosphorous doped silicon semiconductor (doping density: $10^{17}/cm^3$) is heated from 100°C to 200°C. Which one of the following statements is CORRECT?

(a) Position of Fermi level moves towards conduction band

(b) Position of dopant level moves towards conduction band

(c) Position of Fermi level moves towards middle of energy gap

(d) Position of dopant level moves towards middle of energy gap

Ans: (c)

Solution: Phosphorous doped silicon semiconductors behave as a n-type semiconductor. In n-type semiconductor Fermi level lies near conduction band and moves toward middle of the band gap upon heating. At a very high temperature the Fermi level is near the middle of the band gap and semiconductor behaves as intrinsic semiconductor.

Q24. Considering the BCS theory of superconductors, which one of the following statements is **NOT CORRECT**? (h is the Plank's constant and e is the electronic charge)

(a) Presence of energy gap at temperature below the critical temperature

(b) Different critical temperature for isotopes

(c) Quantization of magnetic flux in superconduction ring in the unit of $\left(\frac{h}{e}\right)$

(d) Presence of Meissner effect

Ans: (c)

Solution: Quantization of magnetic flux in superconduction ring in the unit of $\left(\frac{h}{2e}\right)$

Q25. Group I contains elementary excitations in solids. Group II gives the associated field with these excitations. MATCH the excitations with their associated field and select your answer as per codes given below.

Group I	**Group II**
(P) phonon	(i) photon + lattice vibration
(Q) plasmon	(ii) electron +elastic deformation
(R) polaron	(iii) collective electron oscillations
(S) polariton	(iv) elastic wave

Codes

(a) $(P-iv),(Q-iii),(R-i),(S-ii)$ (b) $(P-iv),(Q-iii),(R-ii),(S-i)$

(c) $(P-i),(Q-iii),(R-ii),(S-iv)$ (d) $(P-iii),(Q-iv),(R-ii),(S-i)$

Ans: (b)

Solution: **Phonon**: Quantum of energy of the elastic wave in solid, produced due to the vibration of atoms in solid.

Plasmon: Quantum of energy of the wave produced due to the oscillation of plasma, which contains charged particles (positive ions and negative electrons or ions).

Polaron: A charge placed in a polarizable medium will be screened. The induced polarization will follow the charge carrier when it is moving through the medium. The carrier together with the induced polarization is considered as one entity, which is called a polaron.

Polariton: A polariton is a quasiparticle resulting from the mixing of a photon with phonon.

Q26. A lattice has the following primitive vector $(in\ \text{Å})$: $\vec{a} = 2(\hat{j} + \hat{k})$, $\vec{b} = 2(\hat{k} + \hat{i})$, $\vec{c} = 2(\hat{i} + \hat{j})$. The reciprocal lattice corresponding to the above lattice is

(a) BCC lattice with cube edge of $\left(\frac{\pi}{2}\right)\text{Å}^{-1}$

(b) BCC lattice with cube edge of $(2\pi)\text{Å}^{-1}$

(c) FCC lattice with cube edge of $\left(\frac{\pi}{2}\right)\text{Å}^{-1}$

(d) FCC lattice with cube edge of $(2\pi)\text{Å}^{-1}$

Ans: (a)

Solution: The reciprocal lattice vectors are

$$\vec{a}* = 2\pi\frac{\vec{b}\times\vec{c}}{\vec{a}\cdot(\vec{b}\times\vec{c})} = \frac{\pi}{2}(-\hat{i}+\hat{j}+\hat{k})\text{Å}^{-1}$$

$$\vec{b}* = 2\pi\frac{\vec{c}\times\vec{a}}{\vec{a}\cdot(\vec{b}\times\vec{c})} = \frac{\pi}{2}(\hat{i}-\hat{j}+\hat{k})\text{Å}^{-1}$$

$$\vec{c}* = 2\pi\frac{\vec{a}\times\vec{b}}{\vec{a}\cdot(\vec{b}\times\vec{c})} = \frac{\pi}{2}(\hat{i}+\hat{j}-\hat{k})\text{Å}^{-1}$$

Q27. The total energy of an ionic solid is given by an expression $E = -\frac{\alpha e^2}{4\pi\varepsilon_0 r} + \frac{B}{r^9}$ where α is Madelung constant, r is the distance between the nearest neighbours in the crystal and B is a constant. If r_0 is the equilibrium separation between the nearest neighbours then the value of B is

(a) $\frac{\alpha e^2 r_0^8}{36\pi\varepsilon_0}$ (b) $\frac{\alpha e^2 r_0^8}{4\pi\varepsilon_0}$ (c) $\frac{2\alpha e^2 r_0^{10}}{9\pi\varepsilon_0}$ (d) $\frac{\alpha e^2 r_0^{10}}{36\pi\varepsilon_0}$

Ans: (a)

Solution: At $r = r_0$, $\left.\frac{dE}{dr}\right|_{r=r_0} = 0 = \frac{\alpha e^2}{4\pi\varepsilon_0 r_0^2} - \frac{9B}{r_0^{10}} \Rightarrow B = \frac{\alpha e^2 r_0^8}{36\pi\varepsilon_0}$

GATE-2014

Q28. The Miller indices of a plane passing through the three points having coordinates (0, 0, 1) $(1,0,0)\left(\frac{1}{2},\frac{1}{2},\frac{1}{4}\right)$ are

(a) (212) (b) (111) (c) (121) (d) (211)

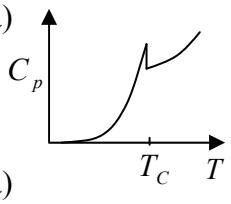
Ans: (a)

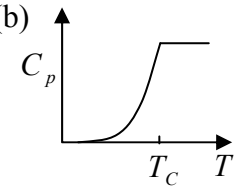
Solution: The equation of plane is determined from following determinant:

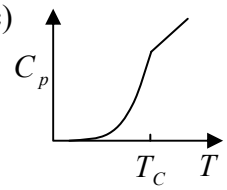
$$\begin{vmatrix} x-1 & y-0 & z-0 \\ -1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{vmatrix} = 0 \Rightarrow (x-1)\left(-\frac{1}{2}\right) - y\left(\frac{-1}{4}+\frac{1}{2}\right) + (z-1)\left(-\frac{1}{2}\right) = 0$$

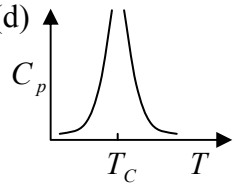
$\Rightarrow \frac{-x}{2} - \frac{y}{4} - \frac{z}{2} + \frac{1}{2} = 0 \Rightarrow 2x + y + 2z - 2 = 0$, $hx + ky + lz - 2 = 0$. Miller indices are $(2\,1\,2)$

Q29. The plot of specifies heat versus temperature across the superconducting transition temperature (T_c) is most appropriately represented by

(a) 

(b) 

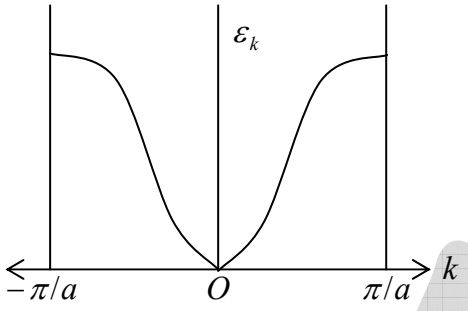
(c) 

(d) 

Ans: (a)

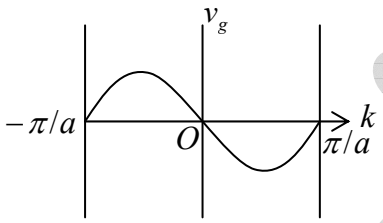
Solution: $C_V \propto e^{-\left(\frac{\Delta}{2kT}\right)}$

Q30. The energy ε_k for band electrons as a function of the wave vector k in the first Brillouin zone $\left(-\frac{\pi}{a} \le k \le \frac{\pi}{a}\right)$ of a one dimensional monoatomic lattice is shown as (a is lattice constant)

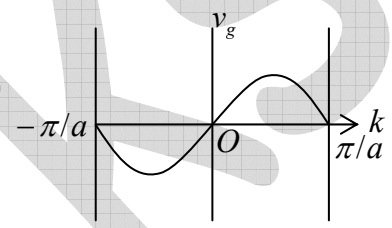


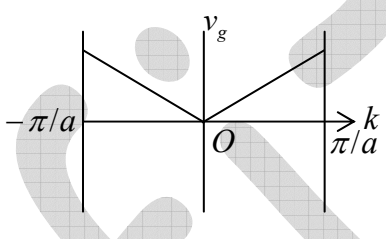
The variation of the group velocity v_g is most appropriately represented by

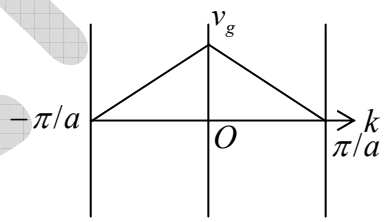
(a)



(b)







Ans: (b)

Solution: $E = \left(E_0 - \gamma\beta\left(\cos ka\right)\right)$

$$V_g = \frac{1}{\hbar}\frac{dE}{dk} = \frac{a\gamma\beta}{\hbar}\sin ka$$

Q31. For Nickel the number density is $8\times10^{23}\ atoms/cm^3$ and electronic configuration is $1s^2 2s^2 2p^6 3s^2 3p^6 3d^8 4s^2$. The value of the saturation magnetization of Nickel in its ferromagnetic state is _____$\times10^9\ A/m$.

(Given the value of Bohr magneton $\mu_B = 9.21\times10^{-21}\ Am^2$)

Ans: 4.42

Solution: Component of magnetic dipoles in a solid material are in the direction of external field.

$M_S =$ (Magnetic dipole moment) $\mu_B N = 0.6\times 9.21\times 10^{-21}\left(Am^2\right)N$.

(For N_i : *magnetic dipole moment* $= 0.6$, Fe :2.22, *For* Cu :1.2)

$N = \dfrac{\rho N_A}{A_N} = 8\times 10^{29}/m^3$, $\mu_B = 9.21\times 10^{-21}\ A/m$

$M_S = 0.6\times 9.21\times 10^{-21}\times 8\times 10^{29} = 4.42\times 10^9\ A/m$, $A_n \rightarrow$ atomic weight

GATE-2015

Q31. The energy dependence of the density of states for a two dimensional non-relativistic electron gas is given by, $g(E) = CE^n$, where C is constant. The value of n is_____

Ans.: 0

Solution: We know that

$g(E) \propto E^{1/2}$ for $3-D$, $g(E) \propto E^0$ for $2-D$, $g(E) \propto E^{-1/2}$ for $1-D$

$\Rightarrow n = 0$ for $2-D$

Q32. The lattice parameters a, b, c of an orthorhombic crystal are related by $a = 2b = 3c$. In units of a the interplanar separation between the (110) planes is _____. (Upto three decimal places)

Ans.: 0.447

Solution: $d_{hkl} = \dfrac{1}{\sqrt{\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}+\dfrac{l^2}{c^2}}} \Rightarrow d_{110} = \dfrac{1}{\sqrt{\dfrac{1}{a^2}+\dfrac{1}{\left(\frac{a}{2}\right)^2}+0}} = \dfrac{a}{\sqrt{5}} = 0.447 \quad \because a = 2b = 3c$

Q33. The dispersion relation for phonons in a one dimensional monoatomic Bravais lattice with lattice spacing a and consisting of ions of masses M is given by $\omega(k) = \sqrt{\dfrac{2c}{M}\left[1-\cos(ka)\right]}$, where ω is the frequency of oscillation, k is the wavevector and C is the spring constant. For the long wavelength modes $(\lambda >> a)$, the ratio of the phase velocity to the group velocity is_____

Ans.: 1

Solution: $\omega(k) = \sqrt{\dfrac{2C}{M}\left[1-\cos(ka)\right]}$

For long wavelength modes $(\lambda >> a)$

$$\because \cos(ka) \cong 1-\frac{(ka)^2}{2} \Rightarrow \omega(k) = \sqrt{\frac{2C}{M}\left[1-1+\frac{(ka)^2}{2}\right]} = a\sqrt{\frac{C}{M}}k$$

Phase velocity $v_P = \frac{\omega}{k} = a\sqrt{\frac{C}{M}}$ and Group velocity $v_g = \frac{d\omega}{dk} = a\sqrt{\frac{C}{M}} \Rightarrow \frac{v_P}{v_g} = 1$

Q34. In a Hall effect experiment, the hall voltage for an intrinsic semiconductor is negative. This is because (symbols carry usual meaning)

(a) $n \approx p$ (b) $n > p$ (c) $\mu_n > \mu_h$ (d) $m_h^* > m_n^*$

Ans.: (c)

Solution: The Hall voltage is $V_H = R_H JB$

where J : current density, B : magnetic field and R_H : Hall constant

$$R_H = \frac{1}{e}\frac{p\mu_p^2 - n\mu_n^2 + (p-n)\mu_n^2\mu_p^2B^2}{(n\mu_n + p\mu_p)^2 + (p-n)^2\mu_n^2\mu_p^2B^2}$$

For intrinsic semiconductor $(n = p = n_i)$ $R_H = \frac{1}{en_i}\frac{\mu_p - \mu_n}{\mu_p + \mu_n}$

In Intrinsic semiconductor $\mu_n > \mu_p$, therefore Hall voltage is negative.

Q35. Given that the Fermi energy of gold is 5.54 eV, the number density of electrons is _____ $\times 10^{28} m^{-3}$ (upto one decimal place)

(Mass of electron $= 9.11\times10^{-31}\ kg$; $h = 6.626\times10^{-34}\ j\cdot s$; $1\ eV = 1.6\times10^{-19}\ j$)

Ans.: 5.9

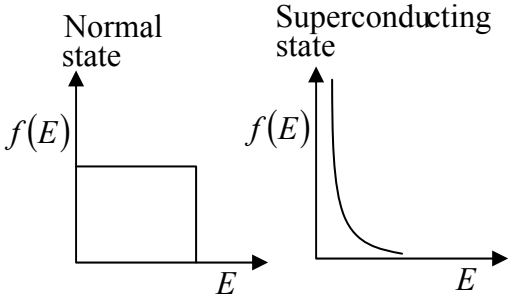
Solution: Relation between electron density (n) and Fermi energy (E_F) is

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \Rightarrow n = \frac{1}{3\pi^2}\frac{(2m)^{3/2}}{\hbar^3}\cdot E_F^{3/2}$$

$$\Rightarrow n = \frac{1}{3\times(3.14)^2}\times\frac{(2\times9.1\times10^{-31}kg)^{3/2}}{(1.0546\times10^{-34}J\text{-}s)^3}\times(5.54\times1.6\times10^{-19}\ \ J)^{3/2}$$

$$\Rightarrow n = \frac{1}{29.61}\times\frac{2.45\times10^{-45}\times8.35\times10^{-28}}{1.17\times10^{-102}}m^{-3} = 0.59\times10^{29}\ m^{-3} \Rightarrow n = 5.9\times10^{28}\ \ m^{-3}$$

Q36. Which one of the following represents the electron occupancy for a superconductor in its normal and superconducting states?

(a)


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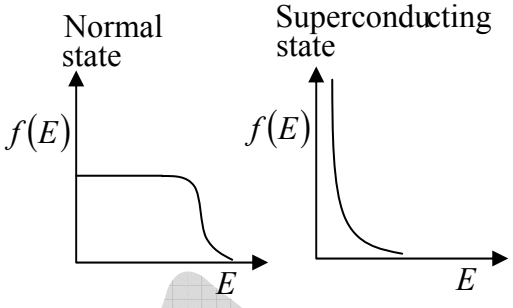
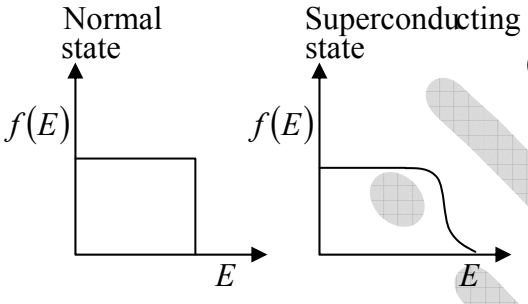
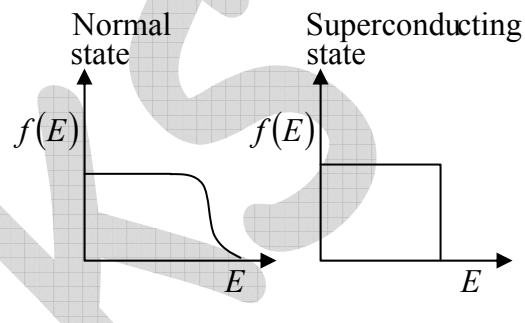


(c)


(d)


Ans. : (d)

Solution: In normal slide, some states below Fermi levels are empty and equal number of states above Fermi levels are filled. If material converts into a superconductor, electrons above the Fermi Level makes cooper pair and they fall back below level Fermi level as same energy released during cooper pair formation. Therefore, correct option is (d).

GATE-2016

Q37. Consider a metal which obeys the Sommerfield model exactly. If E_F is the Fermi energy of the metal at $T = 0K$ and R_H is its Hall coefficient, which of the following statements is correct?

(a) $R_H \propto E_F^{\frac{3}{2}}$

(b) $R_H \propto E_F^{\frac{2}{3}}$

(c) $R_H \propto E_F^{\frac{-3}{2}}$

(d) R_H is independent of E_F.

Ans.: (c)

Solution: $R_H = \dfrac{1}{ne}$, where $E_F = \dfrac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \Rightarrow n = \left(\dfrac{2m}{\hbar^2}\right)^{3/2} \cdot \dfrac{\left(E_F\right)^{3/2}}{3\pi^2} \Rightarrow R_H \propto E_F^{-3/2}$

Q38. A one-dimensional linear chain of atoms contains two types of atoms of masses m_1 and m_2 (where $m_2 > m_1$), arranged alternately. The distance between successive atoms is the same. Assume that the harmonic approximation is valid. At the first Brillouin zone boundary, which of the following statements is correct?

(a) The atoms of mass m_2 are at rest in the optical mode, while they vibrate in the acoustical mode.

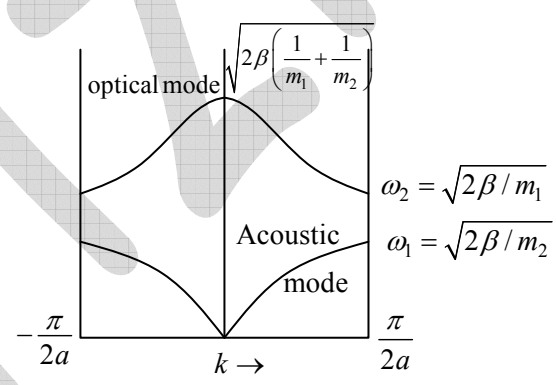
(b The atoms of mass m_1 are at rest in the optical mode, while they vibrate in the acoustical mode.

(c) Both types of atoms vibrate with equal amplitudes in the optical as well as in the acoustical modes.

(d) Both types of atoms vibrate, but with unequal, non-zero amplitudes in the optical as well as in the acoustical modes.

Ans.: (a)

Solution: In optical mode, at Brillouin zone boundary atom of heavier mass (m_2) is at rest, whereas in Acoustic mode, atoms of lighter mass (m_1) is at rest.



Q39. A solid material is found to have a temperature independent magnetic susceptibility, $\chi = C$. Which of the following statements is correct?

(a) If C is positive, the material is a diamagnet.

(b) If C is positive, the material is a ferromagnet.

(c) If C is negative, the material could be a type I superconductor.

(d) If C is positive, the material could be a type I superconductor.

Ans.: (b)

Solution: Susceptibility is defined as $\chi = \frac{\mu}{\mu_0} - 1$, where μ and μ_0 are permeability of medium and vacuum respectively.

(i) For Diamagnet; $\mu < \mu_0$, thus $\chi < 0$ i.e. χ is negative

(ii) For Ferromagnet, $\mu >> \mu_0$, thus $\chi >> 0$ i.e. χ is positive

(iii) For superconductor, $\mu = 0$, thus $\chi = -1$

Thus best answer is (b)

Q40. Atoms, which can be assumed to be hard spheres of radius R, are arranged in an *fcc* lattice with lattice constant a, such that each atom touches its nearest neighbours. Take the center of one of the atoms as the origin. Another atom of radius r (assumed to be hard sphere) is to be accommodated at a position $\left(0, \frac{a}{2}, 0\right)$ without distorting the lattice. The maximum value of $\frac{r}{R}$ is _____. (Give your answer upto two decimal places)

Ans.: 0.41

Solution: The new atom location is $\left(0, \frac{a}{2}, 0\right)$ i.e. it is on the middle of y - axis.

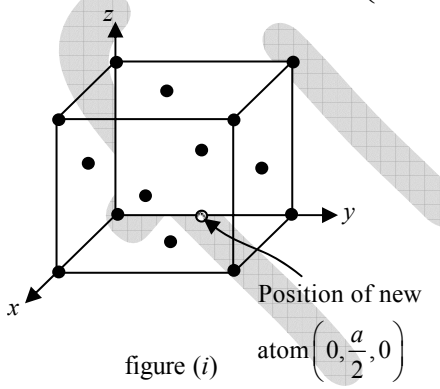


figure (*i*)

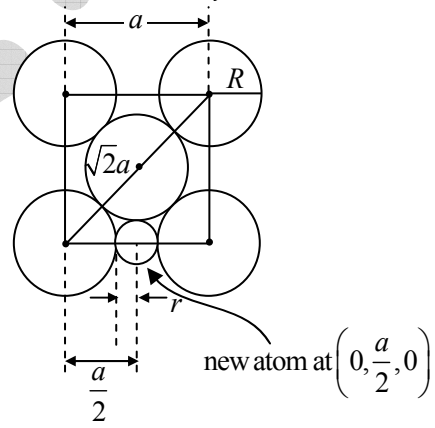


figure (*ii*)

If new atom of radius r fit without distorting the original lattice, then from figure (ii) we get

$$\frac{a}{2} = R + r \qquad \text{(i)}$$

whereas for FCC $\sqrt{2}a = 4R \Rightarrow a = \frac{4}{\sqrt{2}} R = 2\sqrt{2}R$ (ii)

Thus from (i) and (ii) $\frac{2\sqrt{2}}{2}R = R + r \Rightarrow \left(\sqrt{2}-1\right)R = r \Rightarrow \frac{r}{R} = \sqrt{2}-1 = 1.414-1 = 0.414$

Q41. The energy vs. wave vector $(E-k)$ relationship near the bottom of a band for a solid can be approximated as $E = A(ka)^2 + B(ka)^4$, where the lattice constant $a = 2.1\overset{0}{A}$. The values of A and B are $6.3\times10^{-19}\ J$ and $3.2\times10^{-20}\ J$, respectively. At the bottom of the conduction band, the ratio of the effective mass of the electron to the mass of free electron is _____. (Give your answer upto two decimal places)

(Take $\hbar = 1.05\times10^{-34}\ J-s$, mass of free electron $= 9.1\times10^{-31}\ kg$)

Ans.: 0.22

Solution: $E = A(ka)^2 + B(ka)^4$

$\therefore \frac{\partial E}{\partial k} = 2Aa^2k + 4Ba^4k^3$ and $\frac{\partial^2 E}{\partial k^2} = 2Aa^2 + 12Ba^4k^2$

At the bottom of the band $k = 0$

Thus effective mass $m^* = \frac{\hbar^2}{\partial^2 E/\partial k^2} = \frac{\hbar^2}{2Aa^2} = \frac{\left(1.05\times10^{-34}J\cdot s\right)^2}{2\times6.3\times10^{-19}J\times\left(2.1\times10^{-10}m\right)^2}$

$= \frac{1.1025\times10^{-68}}{55.57\times10^{-39}} = 0.01984\times10^{-29} = 19.84\times10^{-32}kg$

$\therefore \frac{m^*}{m} = \frac{19.84\times10^{-32}kg}{9.1\times10^{-31}kg} = 2.18\times10^{-1} = 0.218 \cong 0.22$

Q42. The Fermi energies of two metals X and Y are $5\,eV$ and $7\,eV$ and their Debye temperatures are $170\ K$ and $340\ K$, respectively. The molar specific heats of these metals at constant volume at low temperatures can be written as $(C_V)_X = \gamma_X T + A_X T^3$ and $(C_V)_Y = \gamma_Y T + A_Y T^3$ where γ and A are constants. Assuming that the thermal effective mass of the electrons in the two metals are same, which of the following is correct?

(a) $\frac{\gamma_X}{\gamma_Y} = \frac{7}{5}, \frac{A_X}{A_Y} = 8$

(b) $\frac{\gamma_X}{\gamma_Y} = \frac{7}{5}, \frac{A_X}{A_Y} = \frac{1}{8}$

(c) $\frac{\gamma_X}{\gamma_Y} = \frac{5}{7}, \frac{A_X}{A_Y} = \frac{1}{8}$

(d) $\frac{\gamma_X}{\gamma_Y} = \frac{5}{7}, \frac{A_X}{A_Y} = 8$

Ans.: (a)

Solution: Heat capacity is defined as $C_V = \gamma T + AT^3$

where $\gamma = \frac{3}{2}Nk_B^2 \cdot \frac{1}{E_F}$ and $A = \frac{12\pi^4}{5}Nk_B \cdot \frac{1}{\theta_D^3} \cong 234Nk_B \cdot \frac{1}{\theta_D^3}$

Thus, $\frac{\gamma_X}{\gamma_Y} = \frac{\frac{3}{2}Nk_B^2 \cdot \frac{1}{E_{F_X}}}{\frac{3}{2}Nk_B^2 \cdot \frac{1}{E_{F_Y}}} = \frac{E_{F_Y}}{E_{F_X}} = \frac{7eV}{5eV} = \frac{7}{5}$

and $\frac{A_X}{A_Y} = \frac{234Nk_B \frac{1}{\theta_{D_X}^3}}{234Nk_B \frac{1}{\theta_{D_Y}^3}} = \left(\frac{\theta_{D_Y}}{\theta_{D_X}}\right)^3 = \left(\frac{340}{170}\right)^3 = (2)^3 = 8$

Thus, $\frac{\gamma_X}{\gamma_Y} = \frac{7}{5}$ and $\frac{A_X}{A_Y} = 8$

GATE-2017

Q43. The atomic mass and mass density of Sodium are 23 and $0.968\,g\,cm^{-3}$, respectively. The number density of valence electrons is..... $\times 10^{22}\,cm^{-3}$. (Up to two decimal places) (Avogadro number, $N_A = 6.022 \times 10^{23}$)

Ans. : 2.54

Solution: $\rho = \frac{n_{eff} \times M}{N_A \times a^3} \Rightarrow n = \frac{n_{eff}}{a^3} = \frac{\rho N_A}{M}$

where $\rho = 0.968 g cm^{-3}$, $N_A = 6.022 \times 10^{23}$, $M = 23g$

$$n = \frac{0.968 \times 6.022 \times 10^{23}}{23} = 2.54 \times 10^{22} cm^{-3}$$

Q44. Consider a one dimensional lattice with a weak periodic potential $U(x) = U_0 \cos\left(\frac{2\pi x}{a}\right)$. The gap at the edge of the Brillouin zone $\left(k = \frac{\pi}{a}\right)$ is:

(a) U_0 (b) $\frac{U_0}{2}$ (c) $2U_0$ (d) $\frac{U_0}{4}$

Ans. : (c)

Solution: $U = U_0 \cos\left(\frac{2\pi}{a}x\right)$

Energy at the edge of Brillouin Zone is $U_t = U_0 \cos\left(\frac{2\pi}{a}.\frac{a}{\pi}\right)$

Energy at the $k = 0$ is $U_b = U_0$ $\quad\therefore$ Band gap $\Delta U = U_t - U_b = -2U_0$

Q45. Consider a 2 - dimensional electron gas with a density of $10^{19}\,m^{-2}$. The Fermi energy of the system is…… eV (up to two decimal places).

($m_e = 9.31\times10^{-31}\,kg$, $h = 6.626\times10^{-34}\,Js$, $e = 1.602\times10^{-19}\,C$)

Ans. : 2.34

Solution: $E_F = \left(\frac{\hbar^2}{2m}\right)(2\pi n) = \frac{\left(1.055\times10^{-34}\,J\cdot s\right)^2}{2\times9.31\times10^{-31}}\times2\times3.142\times10^{19}$

$= 0.3756\times10^{-18}\,J = 0.2345\times10\,eV = 2.34\ eV$

Q46. The real space primitive lattice vectors are $\vec{a}_1 = a\hat{x}$ and $\vec{a}_2 = \frac{a}{2}\left(\hat{x}+\sqrt{3}\hat{y}\right)$. The reciprocal space unit vectors $\vec{b}_1$ and $\vec{b}_2$ for this lattice are, respectively

(a) $\frac{2\pi}{a}\left(\hat{x}-\frac{\hat{y}}{\sqrt{3}}\right)$ and $\frac{4\pi}{a\sqrt{3}}\hat{y}$ (b) $\frac{2\pi}{a}\left(\hat{x}+\frac{\hat{y}}{\sqrt{3}}\right)$ and $\frac{4\pi}{a\sqrt{3}}\hat{y}$

(c) $\frac{2\pi}{a\sqrt{3}}\hat{x}$ and $\frac{4\pi}{a}\left(\frac{\hat{x}}{\sqrt{3}}+\hat{y}\right)$ (d) $\frac{2\pi}{a\sqrt{3}}\hat{x}$ and $\frac{4\pi}{a}\left(\frac{\hat{x}}{\sqrt{3}}-\hat{y}\right)$

Ans. (a)

Solution: $\vec{a}_1 = a\hat{x}$, $\vec{a}_2 = \frac{a}{2}\left(\hat{x}+\sqrt{3}\hat{y}\right)$. Assume, $\vec{a}_3 = \hat{z}$

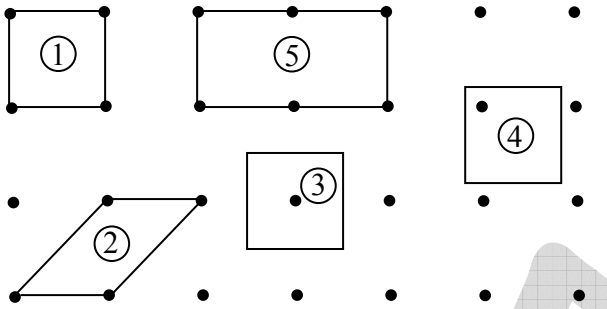
Now, $\vec{a}_1\cdot\left(\vec{a}_2\times\vec{a}_3\right) = a\hat{x}.\left[\frac{a}{2}\left(\hat{x}+\sqrt{3}\hat{y}\right)\times\hat{z}\right] = \frac{a^2\hat{x}}{2}\cdot\left(-\hat{y}+\sqrt{3}\hat{x}\right) = \frac{a^2}{2}\left[-0+\sqrt{3}\right] = \frac{\sqrt{3}a^2}{2}$

$$\therefore \vec{b}_1 = 2\pi\frac{\vec{a}_2\times\vec{a}_3}{\vec{a}_1\cdot\left(\vec{a}_2\times\vec{a}_3\right)} = 2\pi\frac{\left(\sqrt{3}\hat{x}-\hat{y}\right)\frac{a}{2}}{\frac{\sqrt{3}}{2}a^2} = \frac{2\pi}{a}\left(\hat{x}-\frac{\hat{y}}{\sqrt{3}}\right)$$

Similarly, $\vec{b}_2 = 2\pi\frac{\vec{a}_3\times\vec{a}_1}{\vec{a}_1\cdot\left(\vec{a}_2\times\vec{a}_3\right)} = \frac{4\pi}{\sqrt{3}a}\hat{y}$. Thus correct option is (a)

GATE – 2018

Q47. For the given unit cells of a two dimensional square lattice, which option lists all the primitive cells?



(a) (1) and (2) (b) (1), (2) and (3)

(c) (1), (2), (3) and (4) (d) (1), (2), (3), (4) and (5)

Ans. : (c)

Solution: For primitive cell, $N_{eff} = 1$

In cell (1), (2), (3) and (4) $N_{eff} = 1$, these are primitive cell

Whereas in cell (5), $N_{eff} = 2$, this is non-primitive cell.

Q48. At low temperatures (T), the specific heat of common metals is described by (with α and β as constants)

(a) $\alpha T + \beta T^3$ (b) βT^3 (c) $\exp(-\alpha / T)$ (d) $\alpha T + \beta T^5$

Ans. : (a)

Solution: $C = C_e + C_{pn} = \alpha T + \beta T^3$

Q49. The high temperature magnetic susceptibility of solids having ions with magnetic moments can be described by $\chi \propto \frac{1}{T+\theta}$ with T as absolute temperature and θ as constant. The three behaviours i.e., paramagnetic, ferromagnetic and anti-ferromagnetic are described, respectively, by

(a) $\theta < 0,\ \theta > 0,\ \theta = 0$ (b) $\theta > 0,\ \theta < 0,\ \theta = 0$

(c) $\theta = 0,\ \theta < 0,\ \theta > 0$ (d) $\theta = 0,\ \theta > 0,\ \theta < 0$

Ans. : (c)

Solution: Paramagnetism: $\chi = \frac{C}{T}$

Ferromagnetism: $\chi = \frac{C}{T - T_C}$

Anti-ferromagnetism: $\chi = \frac{C}{T + T_C}$

Q50. The energy dispersion for electrons in one dimensional lattice with lattice parameter a is given by $E(k) = E_0 - \frac{1}{2}W\cos ka$, where W and E_0 are constants. The effective mass of the electron near the bottom of the band is

(a) $\frac{2\hbar^2}{Wa^2}$ (b) $\frac{\hbar^2}{Wa^2}$ (c) $\frac{\hbar^2}{2Wa^2}$ (d) $\frac{\hbar^2}{4Wa^2}$

Ans. : (a)

Solution: $E(k) = E_0 - \frac{1}{2}W\cos(ka)$

$$\frac{dE}{dk} = \frac{aW}{2}\sin(ka) \Rightarrow \frac{d^2E}{dk^2} = \frac{a^2W}{2}\cos(ka)$$

$$\therefore\ m^* = \frac{\hbar^2}{\frac{d^2E}{dk^2}} = \frac{\hbar^2}{\frac{a^2W}{2}\cos(ka)} = \frac{2\hbar^2}{Wa^2}$$ [At bottom of the band, $k = 0$]

Q51. Amongst electrical resistivity (ρ), thermal conductivity (κ), specific heat (C), Young's modulus (Y) and magnetic susceptibility (χ), which quantities show a sharp change at the superconducting transition temperature?

(a) ρ, κ, C, Y (b) ρ, C, χ (c) ρ, κ, C, χ (d) κ, Y, χ

Ans. : (b)

GATE-2019

Q52. The relative magnetic permeability of a type-I super conductor is

(a) 0 (b) -1 (c) 2π (d) $\frac{1}{4\pi}$

Ans.: (a)

Solution: $B=\mu_0(H+M)=\mu_0(H+\chi H)=\mu_0(1+\chi)H=\mu H$

$$\therefore \quad \mu=\mu_0(1+x)\Rightarrow \mu_r=\frac{\mu}{\mu_0}=1+\chi$$

For type-I superconductor: $\chi=-1$

$$\therefore \mu_r=1-1=0$$

Q53. In order to estimate the specific heat of phonons, the appropriate method to apply would be

(a) Einstein model for acoustic phonons and Debye model for optical phonons

(b) Einstein model for optical phonons and Debye model for acoustic phonons

(c) Einstein model for both optical and acoustic phonons

(d) Debye model for both optical and acoustic phonons

Ans.: (b)

Solution: At low temperature, the optical branch phonons have energies higher than k_BT and therefore, optical branch waves are not excited. And Debye model is not suitable for optical branch instead it is suitable for acoustical branch. Whereas Einstein model is useful for high temperature and therefore can be applied to optical branch.

Q54. Consider a three-dimensional crystal of N inert gas atoms. The total energy is given by $U(R)=2N\in\left[p\left(\frac{\sigma}{R}\right)^{12}-q\left(\frac{\sigma}{R}\right)^{6}\right]$, where $p=12.13$, $q=14.45$ and R is the nearest neighbour distance between two atoms. The two constants, $\in$ and R, have the dimensions of energy and length, respectively. The equilibrium separation between two nearest neighbour atoms in units of σ (rounded off to two decimal places) is_____

Ans.: 1.09

Solution: $U(R)=2N\in\left[p\left(\frac{\sigma}{R}\right)^{12}-q\left(\frac{\sigma}{R}\right)^{6}\right]$

$$\frac{dU}{dR}=0\Rightarrow 2N\in\left[12p\left(\frac{\sigma}{R}\right)^{11}\cdot\left(\frac{-\sigma}{R^2}\right)-6q\left(\frac{\sigma}{R}\right)^{5}\left(-\frac{\sigma}{R^2}\right)\right]=0$$

$$\Rightarrow 12p\frac{\sigma^{12}}{R^{13}}-6q\frac{\sigma^{6}}{R^{7}}=0$$

$$\Rightarrow 12p\frac{\sigma^{12}}{R^{13}}=6q\frac{\sigma^{6}}{R^{7}}\Rightarrow R^6=\frac{12p}{6q}\sigma^6$$

$$\Rightarrow R=\left(\frac{2p}{q}\right)^{1/6}\sigma \qquad \text{given } p=12.13,\ q=14.45$$

$$\therefore R=\left(\frac{2\times 12.13}{14.45}\right)^{1/6}\sigma=(1.679)^{1/6}\sigma=1.09\sigma$$

Thus $\frac{R}{\sigma}=1.09$

Q55. The energy-wavevector $(E-k)$ dispersion relation for a particle in two dimensions is $E=Ck$, where C is a constant. If its density of states $D(E)$ is proportional to E^p then the value of p is_____

Ans.: 1

Solution: For $E(k)\propto k^s$. The density of states in d - dimension is $D(E)\propto E^{\left(\frac{d}{s}-1\right)}$

Given, $E=Ck$ $\quad\therefore\quad s=1, d=2$

Thus $D(E)\propto E^{\left(\frac{2}{1}-1\right)}$

$\propto E^1$

Q56. A conventional type-I superconductor has a critical temperature of $4.7\,K$ at zero magnetic field and a critical magnetic field of 0.3 Tesla at $0\,K$. The critical field in Tesla at $2K$ (rounded off to three decimal places) is_____

Ans.: 0.246

Solution: $H_c(T) = H_0\left[1-\left(\frac{T}{T_c}\right)^2\right] = 0.3\left[1-\left(\frac{2}{4.7}\right)^2\right] = 0.3\left[1-(0.426)^2\right]$

$= 0.3[1-0.181] = 0.3\times 0.819 = 0.246\, Atm$

Q57. A particle of mass m moves in a lattice along the x - axis in a periodic potential $V(x) = V(x+d)$ with periodicity d. The corresponding Brillouin zone extends from $-k_0$ to k_0 with these two k - points being equivalent. If a weak force F in the x - direction is applied to the particle, it starts a periodic motion with the time period T. Using the equation of motion $F = \frac{dp_{\text{crystal}}}{dt}$ for a particle moving in a band, where p_{crystal} is the crystal momentum of the particle, the period T is found to be (h is Planck constant)

(a) $\sqrt{\frac{2md}{F}}$ (b) $2\sqrt{\frac{2md}{F}}$ (c) $\frac{2h}{Fd}$ (d) $\frac{h}{Fd}$

Ans. : (d)

Solution: $\Delta E = E\int_0^d Fdx = F[x]_0^d = Fd$

Using Heisenberg uncertainty $\Delta E\cdot\Delta t = h$, $T = \Delta t = \frac{h}{\Delta E} = \frac{h}{Fd}$. Thus correct option is (d)

Q58. In a certain two-dimensional lattice, the energy dispersion of the electrons is

$$\varepsilon(\vec{k}) = -2t\left[\cos k_x a + 2\cos\frac{1}{2}k_x a\cos\frac{\sqrt{3}}{2}k_y a\right]$$

where $\vec{k} = (k_x, k_y)$ denotes the wave vector, a is the lattice constant and t is a constant in units of eV. In this lattice the effective mass tensor m_{ij} of electrons calculated at the center of the Brillouin zone has the form $m_{ij} = \frac{\hbar^2}{ta^2}\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$. The value of α (rounded off to two decimal places) is _____

Ans.: 0.333

Solution: Effective mass tensor matrix 4

$$m_{ij} = \begin{bmatrix} \frac{1}{m_{xx}} & \frac{1}{m_{xy}} \\ \frac{1}{m_{yx}} & \frac{1}{m_{yy}} \end{bmatrix} = \begin{bmatrix} \frac{1}{m_{xx}} & 0 \\ 0 & \frac{1}{m_{yy}} \end{bmatrix}$$

When $m_{xx} = \dfrac{\hbar^2}{\partial^2 E / \partial k_x^2}$ and $m_{yy} = \dfrac{\hbar^2}{\partial^2 E / \partial k_y^2}$

Now $\dfrac{\partial E}{\partial k_x} = 2t\left[a\sin k_x a + a\sin\left(\frac{1}{2}k_x a\right)\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$

$$\frac{\partial^2 E}{dk_x^2} = 2t\left[a^2\cos\left(k_x a\right) + \frac{a^2}{2}\cos\left(\frac{1}{2}k_x a\right)\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$$

At the Brillouin zone centre i.e. at $k_x = k_y = 0$

$$\therefore \qquad \frac{\partial^2 E}{\partial k_x^2} = 2ta^2\left(1+\frac{1}{2}\right) = 3ta^2$$

Similarly, $\dfrac{\partial E}{\partial k_y} = 2t\left[\sqrt{3}a\cos\left(\frac{1}{2}k_x a\right)\sin\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$

$$\frac{\partial^2 E}{\partial k_y^2} = 2t\left[\frac{3a^2}{2}\cos\left(\frac{1}{2}k_x a\right)\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$$

At the Brillouin zone centre i.e. at $k_x = k_y = 0$

$$\frac{\partial^2 E}{\partial_y^2} = 3ta^2$$

Thus $m_{xx} = \dfrac{\hbar^2}{\partial^2 E / \partial k_x^2} = \dfrac{\hbar^2}{3ta^2}$ and $m_{yy} = \dfrac{\hbar^2}{\partial^2 E / \partial k_y^2} = \dfrac{\hbar^2}{3ta^2}$

$$m_{ij} = \begin{bmatrix} \frac{\hbar^2}{3ta^2} & 0 \\ 0 & \frac{\hbar^2}{3ta^2} \end{bmatrix} = \frac{\hbar^2}{ta^2}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Thus $\alpha = \dfrac{1}{3} = 0.333$.

NUCLEAR AND PARTICLE PHYSICS

GATE-2010

Q1. The basic process underlying the neutron β-decay is

(a) $d \to u + e^- + \bar{\nu}_e$ (b) $d \to u + e^-$

(c) $s \to u + e^- + \bar{\nu}_e$ (d) $u \to d + e^- + \bar{\nu}_e$

Ans: (a)

Q2. In the nuclear shell model the spin parity of ${}^{15}_{7}N$ is given by

(a) $\frac{1^-}{2}$ (b) $\frac{1^+}{2}$ (c) $\frac{3^-}{2}$ (d) $\frac{3^+}{2}$

Ans: (a)

Solution: $Z = 7$; $(s_{1/2})^2\ (p_{3/2})^4\ (p_{1/2})^1$ and $N = 8$

$$l = 1,\ J = \frac{1}{2} \Rightarrow \text{parity} = (-1)^1 = -1, \quad \text{spin - parity} = \left(\frac{1}{2}\right)^-$$

Q3. Match the reactions on the left with the associated interactions on the right.

(1) $\pi^+ \to \mu^+ + \nu_\mu$ (i) Strong

(2) $\pi^0 \to \gamma + \gamma$ (ii) Electromagnetic

(3) $\pi^0 + n \to \pi^- + p$ (iii) Weak

(a) (1, iii), (2, ii), (3, i) (b) (1, i), (2, ii), (3, iii)

(c) (1, ii), (2, i), (3, iii) (d) (1, iii), (2, i), (3, ii)

Ans: (a)

Q4. The ground state wavefunction of deuteron is in a superposition of s and d states. Which of the following is NOT true as a consequence?

(a) It has a non-zero quadruple moment

(b) The neutron-proton potential is non-central

(c) The orbital wavefunction is not spherically symmetric

(d) The Hamiltonian does not conserve the total angular momentum

Ans: (d)

Q5. The first three energy levels of $^{228}Th_{90}$ are shown below

4^+ ——————— $187keV$
2^+ ——————— $57.5keV$
0^+ ——————— $0keV$

The expected spin-parity and energy of the next level are given by

(a) (6^+; 400 keV) (b) (6^+; 300 keV) (c) (2^+; 400 keV) (d) (4^+; 300 keV)

Ans: (a)

Solution: $\frac{E_2}{E_1}=\frac{J_2(J_2+1)}{J_1(J_1+1)}\Rightarrow\frac{E_6}{E_4}=\frac{6(6+1)}{4(4+1)}\Rightarrow E_6=393keV$

GATE-2011

Q6. The semi-empirical mass formula for the binding energy of nucleus contains a surface correction term. This term depends on the mass number A of the nucleus as

(a) $A^{-1/3}$ (b) $A^{1/3}$ (c) $A^{2/3}$ (d) A

Ans: (c)

Q7. According to the single particles nuclear shell model, the spin-parity of the ground state of $^{17}_{8}O$ is

(a) $\frac{1}{2}^-$ (b) $\frac{3}{2}^-$ (c) $\frac{3}{2}^+$ (d) $\frac{5}{2}^+$

Ans: (d)

Solution: $Z=8$ and $N=9$; $(s_{1/2})^2\ (p_{3/2})^4\ (p_{1/2})^2\ (d_{5/2})^1$

$l=2,\ J=\frac{5}{2}\Rightarrow$ parity $=(-1)^2=+1,$ spin - parity $=\left(\frac{5}{2}\right)^+$

Q8. In the β-decay of neutron $n\rightarrow p+e^-+\bar{\nu}_e$, the anti-neutrino $\bar{\nu}_e$, escapes detection. Its existence is inferred from the measurement of

(a) energy distribution of electrons (b) angular distribution of electrons
(c) helicity distribution of electrons (d) forward-backward asymmetry of electrons

Ans: (a)

Q9. The isospin and the strangeness of Ω^- baryon are

(a) 1, -3 (b) 0, -3 (c) 1, 3 (d) 0, 3

Ans: (b)

GATE-2012

Q10. Deuteron has only one bound state with spin parity 1^+, isospin 0 and electric quadrupole moment 0.286 efm^2. These data suggest that the nuclear forces are having

(a) only spin and isospin dependence

(b) no spin dependence and no tensor components

(c) spin dependence but no tensor components

(d) spin dependence along with tensor components

Ans: (d)

Q11. The quark content of Σ^+, K^-, π^- *and* p is indicated:

$$|\Sigma^+\rangle = |uus\rangle; |K^-\rangle = |s\bar{u}\rangle; |\pi^-\rangle = |\bar{u}d\rangle; |p\rangle = |uud\rangle.$$

In the process, $\pi^- + p \to K^- + \Sigma^+$, considering strong interactions only, which of the following statements is true?

(a) The process, is allowed because $\Delta S = 0$

(b) The process is allowed because $\Delta I_3 = 0$

(c) The process is not allowed because $\Delta S \neq 0$ and $\Delta I_3 \neq 0$

(d) The process is not allowed because the baryon number is violated

Ans: (c)

Solution: $\pi^- + p \to k^- + \sum^+$

S: $\quad 0 \quad 0 \quad -1 \quad -1$ (not conserved)

I_3: $\quad -1 \quad +\frac{1}{2} \quad -\frac{1}{2} \quad +1$ (not conserved)

For strong interaction S and I_3 must conserve. Therefore this process is not allowed under strong interaction

Q12. Which one of the following sets corresponds to fundamental particles?

(a) proton, electron and neutron

(b) proton, electron and photon

(c) electron, photon and neutrino

(d) quark, electron and meson

Ans: (a)

Q13. In case of a Geiger-Muller (GM) counter, which one of the following statement is CORRECT?

(a) Multiplication factor of the detector is of the order of 10^{10}

(b) Type of the particles detected can be identified

(c) Energy of the particles detected can be distinguished

(d) Operating voltage of the detector is few tens of Volts

Ans: (c)

Q14. Choose the CORRECT statement from the following

(a) Neutron interacts through electromagnetic interaction

(b) Electron does not interact through weak interaction

(c) Neutrino interacts through weak and electromagnetic interaction

(d) Quark interacts through strong interaction but not through weak interaction

Ans: (d)

GATE-2013

Q15. The decay process $n \to p^+ + e^- + \bar{\nu}_e$ violates

(a) Baryon number (b) lepton number (c) isospin (d) strangeness

Ans: (c)

Q16. The isospin (I) and baryon number (B) of the up quark is

(a) $I = 1, B = 1$ (b) $I = 1, B = 1/3$

(c) $I = 1/2, B = 1$ (d) $I = 1/2, B = 1/3$

Ans: (d)

Q17. In the β decay process, the transition $2^+ \to 3^+$, is

(a) allowed both by Fermi and Gamow-Teller selection rule

(b) allowed by Fermi and but not by Gamow-Teller selection rule

(c) not allowed by Fermi but allowed by Gamow-Teller selection rule

(d) not allowed both by Fermi and Gamow-Teller selection rule

Ans: (c)

Solution: According to Fermi Selection Rule:

$\Delta I = 0, \quad Parity = No\ Change$

According to Gammow-Teller Selection Rule:

$\Delta I = 0, \pm 1, \quad Parity = No\ Change$

In the β decay process, the transition $2^+ \to 3^+$,

$\Delta I = \pm 1, \quad Parity = No\ Change$.

GATE-2014

Q18. Which one of the following is a fermions'?

(a) α -particle (b) ${}_4Be^7$ nucleus

(c) Hydrogen atom (d) deuteron

Ans: (b)

Solution: If a nucleus contains odd number of nucleons, it is fermions. If a nucleus contains even number of nucleons, it is a boson.

Q19. Which one of the following three-quark states (qqq) denoted by X CANNOT be a possible baryon? The corresponding electric charge is indicated in the superscript.

(a) X^{++} (b) X^{+} (c) X^{-} (d) X^{--}

Ans: (d)

Solution: $X = qqq$

$X^{++}(uuu)\frac{2}{3}+\frac{2}{3}+\frac{2}{3}=\frac{6}{3}=2(\text{two unit positive charge})$

$X^{+}(uud)\frac{2}{3}+\frac{2}{3}-\frac{1}{3}=\frac{4}{3}-\frac{1}{3}=1(\text{single unit positive charge})$

$X^{-}(ddd)=-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}=-1(\text{single unit negative charge})$

$X^{--}[\text{Not possible with } qqq]$. So the correct option is (d)

Q20. Consider the process $\mu^+ + \mu \to \pi^+ + \pi^-$. The minimum kinetic energy of the muons (μ) in the centre of mass frame required to produce the pion (π) pairs at rest is _____ MeV .

Ans: 81.7

Solution: Use conservation of energy and momentum in relativistic form.

$$m_\mu = 105\ MeV/c^2 \quad and \quad m_\pi = 140\ MeV/c^2$$

$$E_\mu = \frac{\left(m_{\pi^+} + m_{\pi^-}\right)^2 c^2 - \left(m_{\mu^+} + m_\mu\right)^2 c^2}{2m_\mu} \Rightarrow E_\mu = \frac{(280)^2\ MeV - (210)^2\ MeV}{2\times 105} = 163.3\ MeV$$

For pair it will be $\frac{163.3}{2}\ MeV = 81.7\ MeV$

Q21. A nucleus X undergoes a first forbidden β-decay to nucleus Y. If the angular momentum (I) and parity (P), denoted by I^P as $\frac{7^-}{2}$ for X, which of the following is a possible I^P value for Y?

(a) $\frac{1^+}{2}$ (b) $\frac{1^-}{2}$ (c) $\frac{3^+}{2}$ (d) $\frac{3^-}{2}$

Ans: (c)

Solution: For first forbidden β-decay; $\Delta I = 0,1$ or 2 and Parity does change.

GATE-2015

Q22. The decay $\mu^+ \to e^+ + \gamma$ is forbidden, because it violates

(a) momentum and lepton number conservations

(b) baryon and lepton number conservations

(c) angular momentum conservation

(d) lepton number conservation

Ans.: (d)

Solution: $\mu^+ \to e^+ + \gamma$. In this decay lepton number is not conserved.

Q23. A beam of X - ray of intensity I_0 is incident normally on a metal sheet of thickness $2\ mm$. The intensity of the transmitted beam is $0.025\ I_0$. The linear absorption coefficient of the metal sheet $(in\ m^{-1})$ is _____ (upto one decimal place)

Ans.: 1844.4

Solution: $I = I_0 e^{-\mu x} \Rightarrow \mu = \frac{1}{x}\ln\left(\frac{I_0}{I}\right) = \frac{1}{2\times 10^{-3}}\ln\left(\frac{I_0}{0.025 I_0}\right) = \frac{1}{2\times 10^{-3}}\ln(40)$

$$\Rightarrow \mu = \frac{2.303}{2\times 10^{-3}}\left[\log_{10} 40\right] = 1.151\times 10^3\left[2\times 0.3010 + 1\right] = 1844.4\ m^{-1}$$

Q24. The mean kinetic energy of a nucleon in a nucleus of atomic weight A varies as A^n, where n is_____(upto two decimal places)

Ans.: -0.67

Solution: $\langle T\rangle = \dfrac{\int_0^R -\dfrac{\hbar^2}{2m}\left(\dfrac{d^2}{dr^2}+\dfrac{1}{r}\dfrac{d}{dr}\right)4\pi r^2 dr}{\int_0^R 4\pi r^2 dr} = \dfrac{-\dfrac{\hbar^2}{2m}4\pi\int_0^R (2+2)\,dr}{\int_0^R 4\pi r^2 dr} = \dfrac{-\dfrac{\hbar^2}{2m}4\pi\times 4R}{4\pi R^3/3}$

$$\Rightarrow \langle T\rangle \propto \frac{1}{R^2} = \frac{1}{\left(R_0 A^{\frac{1}{3}}\right)^2} = \frac{1}{A^{\frac{2}{3}}} = A^{-\frac{2}{3}} \Rightarrow n = -\frac{2}{3} = -0.667 = -0.67$$

Q25. The atomic masses of ${}^{152}_{63}Eu$, ${}^{152}_{62}Sm$, ${}^{1}_{1}H$ and neutron are 151.921749, 151.919756, 1.007825 and 1.008665 in atomic mass units (amu), respectively. Using the above information, the Q- value of the reaction ${}^{152}_{63}Eu + n \rightarrow {}^{152}_{62}Sm + p$ is _____ $\times 10^{-3}$ amu (upto three decimal places)

Ans.: 2.833

Solution: $Q = 152.930414 - (152.927581) = 2.833\times 10^{-3}\, a.m.u.$

Q26. In the nuclear shell model, the potential is modeled as $V(r) = \dfrac{1}{2}m\omega^2 r^2 - \lambda\vec{L}\cdot\vec{S},\ \lambda > 0$. The correct spin-parity and isospin assignments for the ground state of ${}^{13}_{6}C$ is

(a) $\dfrac{1^-}{2};\dfrac{-1}{2}$ (b) $\dfrac{1^+}{2};\dfrac{-1}{2}$ (c) $\dfrac{3^+}{2};\dfrac{1}{2}$ (d) $\dfrac{3^-}{2};\dfrac{-1}{2}$

Ans.: (a)

Solution: ${}^{13}C_6$, $N = 7, Z = 6$, for $N = 7$; $\left(1S_{\frac{1}{2}}\right)^2\left(1P_{\frac{3}{2}}\right)^4\left(P_{\frac{1}{2}}\right)^1 \Rightarrow j = \dfrac{1}{2}\ and\ \ l = 1$

Thus spin- parity is $\left(\dfrac{1}{2}\right)^-$.

GATE-2016

Q27. In the $SU(3)$ quark model, the triplet of mesons (π^+, π^0, π^-) has

(a) Isospin $= 0$, Strangeness $= 0$ (b) Isospin $= 1$, Strangeness $= 0$

(c) Isospin $= \frac{1}{2}$, Strangeness $= +1$ (d) Isospin $= \frac{1}{2}$, Strangeness $= -1$

Ans.: (b)

Solution: π^+, π^0, π^- are not strange particle thus strangness $= 0$

Since meson group contain 3 particles, thus $I = 1$

Q28. Consider the reaction ${}^{54}_{25}Mn + e^- \rightarrow {}^{54}_{24}Cr + X$. The particle X is

(a) γ (b) ν_e (c) n (d) π^0

Ans.: (b)

Q29. Which of the following statements is NOT correct?

(a) A deuteron can be disintegrated by irradiating it with gamma rays of energy $4\ MeV$.

(b) A deuteron has no excited states.

(c) A deuteron has no electric quadrupole moment.

(d) The ${}^1S_0$ state of deuteron cannot be formed.

Ans.: (c)

Q30. According to the nuclear shell model, the respective ground state spin-parity values of ${}^{15}_{8}O$ and ${}^{17}_{8}O$ nuclei are

(a) $\frac{1^+}{2}, \frac{1^-}{2}$ (b) $\frac{1^-}{2}, \frac{5^+}{2}$ (c) $\frac{3^-}{2}, \frac{5^+}{2}$ (d) $\frac{3^-}{2}, \frac{1^-}{2}$

Ans.: (b)

Solution: ${}^{15}_{8}O$; $Z = 8$ and $N = 7$; $N = 7 : (s_{1/2})^2 (p_{3/2})^4 (p_{1/2})^1$

$\Rightarrow j = \frac{1}{2}$ and $l = 1$. Thus spin and parity $= \left(\frac{1}{2}\right)^-$

${}^{17}_{8}O$; $Z = 8$ and $N = 9$; $N = 9 : (s_{1/2})^2 (p_{3/2})^4 (p_{1/2})^2 (d_{5/2})^1$

$\Rightarrow j = \frac{5}{2}$ and $l = 2$. Thus spin and parity $= \left(\frac{5}{2}\right)^+$

GATE-2017

Q31. Which one of the following conservation laws is violated in the decay $\tau^+ \to \mu^+\mu^+\mu^-$

(a) Angular momentum
(b) Total Lepton number
(c) Electric charge
(d) Tau number

Ans. : (d)

Solution:

$$\tau^+ \to \mu^+ + \mu^+ + \mu^-$$

$q = +1 \quad +1 \quad +1 \quad -1$ conserved

$L = +1 \quad +1 \quad +1 \quad -1$ conserved

$L_\tau = +1 \quad 0 \quad 0 \quad 0$ Not conserved

spin $= 1/2 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}$ conserved

Tau number is not conserved

Q32. Electromagnetic interactions are:

(a) C conserving
(b) C non-conserving but CP conserving
(c) CP non-conserving but CPT conserving
(d) CPT non-conserving

Ans. : (a)

Solution: In electromagnetic interaction C is conserved

CPT: Conserved in all interaction

CP: Conserved in EM and Strong interactions

$$E_n = \frac{-13.6}{n^2}(eV)$$

For $n = 1$, $E_1 = -13.6\ eV$ Ground state

For $n = \infty$, $E_\infty = 0$ Highest state

Thus, correct option is (a)

Q33. In the nuclear reaction ${}^{13}C_6 + \nu_e \to {}^{13}N_7 + X$, the particle X is

(a) an electron
(b) an anti-electron
(c) a muon
(d) a pion

Ans. : (a)

Solution: $^{13}C_6 + \nu_e \rightarrow {}^{13}N_7 + X \Rightarrow {}^{13}C_6 \rightarrow {}^{13}N_7 + X + \bar{\nu}_e$

$L_e = 0 \qquad 0 \quad +1 \quad -1$

To conserve the Lepton number L_e, X should be e^-

Q34. J^P for the ground state of the $^{13}C_6$ nucleus is

(a) 1^+ (b) $\frac{3^-}{2}$ (c) $\frac{3^+}{2}$ (d) $\frac{1^-}{2}$

Ans. : (d)

Solution: $^{13}C_6$: $Z = 6,\ N = 7,\ N = 7$: $(s_{1/2})^2 (p_{3/2})^4 (p_{1/2})^1$

$\Rightarrow j = \frac{1}{2}$ and $l = 1$. Thus spin and parity $= \left(\frac{1}{2}\right)^-$

Q35. The π^+ decays at rest to μ^+ and ν_μ. Assuming the neutrino to be massless, the momentum of the neutrino is…….. MeV/c. (up to two decimal places)

$(m_\pi = 139\,MeV/c^2, m_\mu = 105\,MeV/c^2)$

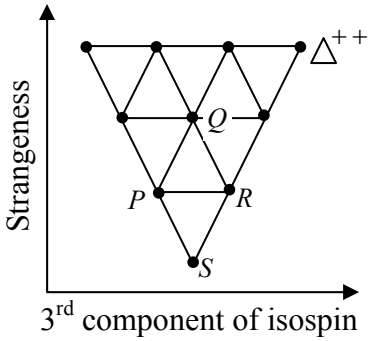
Ans. : 29.84

Solution: $E_\nu = \frac{\left(m_\pi^2 - m_\mu^2\right)c^2}{2m_\pi} = p \times c$

So $p = \frac{\left(m_\pi^2 - m_\mu^2\right)c}{2m_\pi} = \frac{19321 - 11025}{2 \times 139\,c} = \frac{29.84}{c}(MeV)$

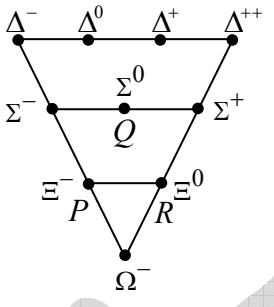
GATE-2018

Q36. The elementary particle Ξ^0 is placed in the baryon decuplet, shown below, at



(a) P (b) Q (c) R (d) S

Ans. : (c)



Q37. In the decay, $\mu^+ \to e^+ + \nu_e + X$, what is X ?

(a) γ (b) $\bar{\nu}_e$ (c) ν_μ (d) $\bar{\nu}_\mu$

Ans. : (d)

Solution:- $u^+ \to e^+ + \nu_e + \bar{\nu}_u$

L_u : $-1 \quad 0 \quad 0 \quad -1$

L_e : $0 \quad -1 \quad +1 \quad 0$

Q38. For nucleus ^{164}Er, a $J^\pi = 2^+$ state is at 90 keV. Assuming ^{164}Er to be a rigid rotor, the energy of its 4^+ state is _____ keV (up to one decimal place)

Ans. : 300

Solution: $E_J = hcBJ(J+1)$

$E_{2^+} = hc\ B\ 2(2+1)$ and $E_{4^+} = hc\ B\ 4(4+1)$

_____ 4^+

_____ 2^+

Then, $\dfrac{E_{4^+}}{E_{2^+}} = \dfrac{20}{6} \Rightarrow E_{4^+} = \dfrac{20}{6} \times 90\ keV = 300\,keV$

Q39. Inside a large nucleus, a nucleon with mass $939\,MeVc^{-2}$ has Fermi momentum 1.40 fm^{-1} at absolute zero temperature. Its velocity is Xc, where the value of X is_____ (up to two decimal places).

($\hbar c = 197\,MeV$-fm)

Ans. : 0.29

Solution: Here, fermi – momentum or fermi radius, $k_F = 1.40\,fm^{-1}$ and $\hbar c = 197$ Mev – fm

Now, Fermi velocity –

$$V_F = \frac{P}{m} = \frac{\hbar k_F}{m} = \frac{(\hbar c)k_F \cdot c}{mc_2} = \frac{(197)\times 1\cdot 40\times c}{939} = \frac{275\cdot 8c}{939} = 0.29c$$

Q40. An α particle is emitted by a $^{230}_{90}Th$ nucleus. Assuming the potential to be purely Coulombic beyond the point of separation, the height of the Coulomb barrier is_____ MeV (up to two decimal places).

($\frac{e^2}{4\pi \in_0} = 1.44\,MeV$-fm, $r_0 = 1.30$ fm)

Ans. : 25.995

Solution: The height of coulomb barrier for α particle from

$$_{90}Th^{230} \rightarrow_{88} X^{226} + 2He^4 \ (\alpha \text{ - particle})$$

$$V_C = \frac{1}{4\pi \in_0}\left(\frac{2ze^2}{R}\right)$$

Here, $R_0 = 1.3\ fm$, $\frac{e^2}{4\pi \in_0} = 1.44\ MeV\ fm$

And $R = R_0 A^{1/3}$

Here, we consider pure Coulombic interection

$$A_{Th}^{1/3} = A_X^{1/3} + A_\alpha^{1/3} = (226)^{1/3} + (4)^{1/3} = (6.09 + 1.58) = 7.67$$

$$R = R_0 A_{Th}^{1/3} = 1.3(7.67)$$

Hence, $V_C = \left(\frac{e^2}{4\pi \in_0}\right)\frac{2\times 90}{1.3(7.67)} = \frac{180\times 1.44}{1.3\times 7.67}\frac{MeV}{fm}$

$$V_C = 25.995\ MeV$$

GATE-2019

Q41. Considering baryon number and lepton number conservation laws, which of the following process is/are allowed?

(i) $p \rightarrow \pi^0 + e^+ + \nu_e$

(ii) $e^+ + \nu_e \rightarrow \mu^+ + \nu_\mu$

(a) both (i) and (ii) (b) only (i) (c) only (ii) (d) neither (i) nor (ii)

Ans. : (c)

Solution: (i) $P \rightarrow \pi^0 + e^+ + \nu_e$

$B: +1 \quad 0 \quad 0 \quad 0$: Not conserved

Therefore, this is not an allowed process

(ii) $e^+ \quad +\nu_e \quad \rightarrow \quad \mu^+ + \nu_\mu$

$q: \quad +1 \quad 0 \quad +1 \quad 0$: conserved

$spin: \quad 1/2 \quad 1/2 \quad 1/2 \quad 1/2$: conserved

$L_e: \quad -1 \quad +1 \quad 0 \quad 0$: conserved

$L_\mu: \quad 0 \quad 0 \quad -1 \quad +1$: conserved

Since neutrino is involve, therefore parity is violated. This is allowed through weak interaction

Q42. A massive particle X in free space decays spontaneously into two photons. Which of the following statements is true for X ?

(a) X is charged

(b) Spin of X must be greater than or equal to 2

(c) X is a boson

(d) X must be a baryon

Ans. : (c)

Solution: $X \rightarrow r + r$

$q: \quad 0 \quad 0 \quad 0$

$spin: \quad 0,1,2 \quad 1 \quad 1$

Thus spin of X can be either $0,1$ or 2 . (integer)

Therefore, option (b) is wrong while option (c) is correct.

Q43. The nuclear spin and parity of $^{40}_{20}Ca$ in its ground state is

(a) 0^+ (b) 0^- (c) 1^+ (d) 1^-

Ans.: (a)

Solution: $^{40}_{20}Ca$ is an even-even nuclei, therefore $I = 0, P = +ve$

$\therefore$ Spin-parity $= 0^+$

Q44. Low energy collision (s - wave scattering) of pion (π^+) with deuteron (d) results in the production of two proton ($\pi^+ + d \to p + p$). The relative orbital angular momentum (in units of $\hbar$) of the resulting two-proton system for this reaction is

(a) 0 (b) 1 (c) 2 (d) 3

Ans.: (b)

Solution: $\pi^+ + d \quad \to \quad p + p$

Parity: $(-1)\times(+1) \qquad (-1)^l \ \pi_p \pi_p$

$\therefore \ (-1)^l \ \pi_p \pi_p = -1$

Since $\pi_p = +1 \qquad \therefore \ (-1)^l = -1$

Thus, $l = 1$.

Q45. A radioactive element X has a half-life of 30 hours. It decays via alpha, beta and gamma emissions with the branching ratio for beta decay being 0.75. The partial half-life for beta decay in unit of hours is _____

Ans.: 40

Solution: Branching ratio is the fraction of particles (here β) which decays by an individual decay mode with respect to the total number of particles which decays

$$BR = \frac{\left(\frac{dN}{dt}\right)_x}{\left(\frac{dt}{dt}\right)_\beta} = \frac{(T_{1/2})_x}{(T_{1/2})_\beta} \Rightarrow (T_{1/2})_\beta = \frac{(T_{1/2})_x}{BR} = \frac{30}{0.75} = 40 \text{ hours}$$